Repairable Two Phase Service M^X/G/1 Queueing Models With Infinite Number Of Immediate Feedbacks Under Bernoulli Schedule Vacation

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ABSTRACT: In the present paper the author considers an unreliable M^X/G , $G_{i(1 \le \le c)}/I$ queueing system with two phases of heterogeneous service in which the server operates single service in the first phase and multi-optional heterogeneous service facilities in second phase. The arriving customers have to undergo the first phase service and any one of the second optional services to complete the first round of service. After completing the first round of service, the customers may demand for re-services from the second phase, infinitely many times before leaving the system. The server is subject to unpredictable breakdowns during busy period and sent for repair immediately. During breakdown period, the service interrupted customers stay in the service facility to complete the remaining service. It is further assumed that, after a successful completion of a service (including the feedback services) of each customer the server may take a Bernoulli schedule vacation before starting a new service for the next customer.

KEYWORDS :Batch Arrival, Bernoulli Schedule vacation , Infinite feedback, Multi-Optional services, Unreliable,

I. INTRODUCTION

In feedback queueing models, if the service of a job of a customer is unsuccessful then, the customer tries the job again and again until a successful service is completed. In computer and communication networks with cyclic queueing systems, messages are processed in two stages and a fraction of the messages may re-enter the system. Queueing systems with various feedback policies have been investigated by many authors. The concept of feedback was introduced by Takacs (1963) and since then many papers have appeared about this topic. He considered an M/G/1 Bernoulli feedback queue with single class customers and obtained the distributions of queue size and the total response time of a customer. Disney and Konig (1985) have given an overview of the literature concerning Bernoulli feedback studies. Fewer results are known for feedback queueing systems in which the feedback policy is not Bernoulli. Baskett et al. (1975) obtained the product form of the joint queue size distribution for the M/M/1 queueing system with several types of customers and general feedback policy. Thangaraj and Vanitha (2010), Choi and Tae-Sung (2003), BadamchiZadeh and Shahkar (2008) and Choudhury and Paul (2005) derived the queue size distribution for M/G/1 queue with two phases of heterogeneous services and Bernoulli feedback system. Saravanarajan and Chandrasekaran (2014) analysed MX/G/1 feedback queue with two-phase service, compulsory server vacation and random breakdowns. Kalidass and Kasturi (2013) have considered a reliable Poisson arrival M/G/1 queueing system with two phases of heterogeneous services and finite number (m) of immediate Bernoulli feedbacks. They have assumed that, all the arriving customers are provided with the same type of service in the first phase and allowed to choose one of the optional services from the second phase. After having completed services in both phases, the customer can make an immediate feedback. In the feedback service, the first phase of service is of the same type as in the previous service and in the second phase, the customer is permitted to choose an optional service (may be different from the one chosen earlier). In the present chapter a generalization of their model in the sense that the feedback customers can demand re-services either from the first phase or from the second phase alone is considered for a repairable MX/G/1 queueing system with infinite number of feedbacks under Bernoulli Schedule Vacation. The results of Kalidass and Kasturi are verified under the conditions, $f^2 = 0$, m

 $\rightarrow \infty$ and by setting the breakdown parameters and the vacation control parameter to zero for the single arrival Poisson queue.

1.1 M^X/G/1 QUEUE WITH INFINITE NUMBER OF FEEDBACKS AND RESUMPTION OF INTERRUPTED SERVICE

1.1.1 MATHEMATICAL ANALYSIS OF THE SYSTEM

1.1.1.1 Model Description

The present chapter deals with $M^X/G/1$ queueing system with two-phases of service and Bernoulli vacation schedule for an unreliable server which consists of a breakdown period. The customers arrive at the system in batches of variable size in accordance with time-homogeneous compound Poisson process with group arrival rate λ .

Service is provided one by one according to FCFS basis. Every customer has to undergo two stages of services following different general (arbitrary) distributions. The arriving customers first receive the First Phase Service S (FPS), which is followed by any of the second phase services (SPS) S_i ($1 \le i \le C$). The customers after completing the first phase service, can choose any of the ith optional services (available in second phase) with probability r_i, where $\sum_{i=1}^{C} r_i = 1$. The second phase service commences immediately after the completion of first phase service, and all the services are provided by the same server. A customer is said to complete the first round of service if he undergoes the FPS and anyone of the ith second optional services with probability r_i ($1 \le C$). The first round of service may be termed as primary (or) fresh service.

The customer, who finishes the first round of service either feeds back immediately from the first phase with probability f_1 (or) from any of the ℓ^{th} second optional services with probability $f_2 r_{\ell}$ ($1 \le \ell \le C$) or leaves the system with probability ($1 - (f_1 + f_2)$). After the completion of the first feedback service, the customer may again go in for a third round of service and so on in a similar way. Thus the first round service consists of services of length $S + S_i$ ($1 \le i \le C$). Second round of service (i.e., first feedback service) either is of length $S + S_{\ell}$ with probability f_1 (or) S_{ℓ} alone with probability $f_2 r_{\ell}$ (or) else does not exist with probability ($1 - (f_1 + f_2)$). This process will continue any number of times until the customer is satisfied. The next customer in the queue can go into the system only after the successful completion of all feedback rounds of the preceding customer. The distribution functions, density functions, LST of FPS and SPS are respectively denoted by S(t), $S_i(t)$; s(t), $s_i(t)$, $S^*(\theta)$, $S_i^-(\theta)$, with finite moments.

During busy period, the server is subject to breakdowns. The lifetime of the server follows exponential distribution with parameters : a^0 in the primary FPS ; a in the feedback FPS ; a_i^0 in the primary ith second phase service and a_i in the ith second optional feedback service.

The server whenever breaks down is sent for repair immediately and the customer just being served, waits in the corresponding service facility to complete the remaining service.

The repair time distributions of the server follow arbitrary distributions $R_1^0(t)$, $R_1(t)$, $R_i^0(t)$ and $R_i(t)$ respectively according as the breakdowns occur in first phase due to primary service or feedback service (or) ith primary or feedback services. It is also assumed that after completing a service to a customer (i.e., when the customer leaves the system) the server may take a Bernoulli Scheduled Single Vacation (BSV) with probability **p** or continue to serve the next customer in the queue if any (or) stay idle in the system for the next batch to arrive, with probability (1 - p). The vacation time V follows general distribution with its distribution function V(t), density function v(t), LST $V^*(\theta)$ with finite first and second moments $E(V^k)$, k = 1, 2.

function V(t), density function V(t), LST V(0) with finite first and second moments E(V), K = 1, 2

Thus a cycle consists of primary services, feedback services, breakdown period and vacation period. Various stochastic processes involved in the queueing system are assumed to be independent of each other. The customers continue to arrive and join the system independent of the system states, following the compound Poisson process. Using supplementary variable technique the steady state system equations under the steady-state condition are analysed and the PGF of the system size is obtained so that various performance measures of the model can be derived from it.

The notations of Random Variables (RV), Cumulative Distribution Function (CDF), Probability Density Function (PDF), Laplace-Stieltjes Transform (LST) and its kth moments of the RVs are listed below :

	RV	CDF	PDF	LST	k^{th} moments $k = 1, 2$
Service time in first phase	S	S(x)	s(x)	$S^*(\theta)$	E(S ^k)
i th optional service time of the second phase	Si	S _i (x)	s _i (x)	$S_i^*(\theta)$	$E(S_i^k),$ i = 1 to C
Repair time in first primary service	R_1^0	R ₁ ⁰ (y)	r ₁ ⁰ (y)	$R_{1}^{0^{*1}}(\theta_{1})$	$E((R_{1}^{0})^{k})$
Repair time in first feedback service	R_1	R ₁ (y)	r ₁ (y)	$R_1^{*_1}(\theta_1)$	$E(R_1^k)$
Repair time in second primary service	${\sf R}_2^{(i,0)}$	$R_{2}^{(i,0)}(y)$	$r_{2}^{(i,0)}(y)$	${\sf R}_2^{(i,0)^{*1}}(\theta_1)$	$E((R_{2}^{(i,0)})^{k})$
Repair time in second feedback service	R_2^i	$R_2^i(y)$	$r_2^i(y)$	$R_{2}^{\mathrm{i}^{*1}}(\theta_{1})$	$E((R_2^i)^k)$
Vacation time	V	V(x)	v(x)	V [*] (θ)	$E(V^k)$

Table 1.1.1.

Let $N_{S}(t)$ denote the system size at time t and $S^{\circ}(t)$, $S_{i}^{\circ}(t)$, $(R_{1}^{0})^{\circ}(t)$, $(R_{1})^{\circ}(t)$, $(R_{2}^{(i,0)})^{\circ}(t)$,

 $(R_2^i)^{\circ}(t), V^{\circ}(t)$ respectively denote the remaining times of the random variables ; service time in first stage, second stage, repair time in first fresh service, first feedback service, second fresh service, second feedback service and vacation time at time t. Further the server states are denoted by the random variable Y(t) at time t. Then the state space is $\{N_s(t), \delta(t)\}$ where $\delta(t) = (0, S^{\circ}(t), S_i^{\circ}(t), (R_1^0)^{\circ}(t), (R_2^{(i,0)})^{\circ}(t), (R_2^{(i,0)})^{\circ$

 $PI(t) = Pr \{N_S(t) = 0, Y(t) = 0\}$, when the server is idle.

For $n \ge 1$ and $1 \le i \le C$

$P_{1,n}^{0}(x,t) dt$	=	Pr {N _S (t) = n, x < $S^{o}(t) \le x + dt$, Y(t) = 1}, a customer is being served in the primary first phase service
$P_{1,n}(x, t) dt$	=	Pr {N _S (t) = n, x < $S^{o}(t) \le x + dt$, Y(t) = 1}, a customer is being served in the first phase – feedback service

$P_{2,n}^{(i,0)}(x,t) dt =$	Pr {N _S (t) = n, x < $S_i^o(t) \le x + dt$, Y(t) = 2}, a customer is being served in the i th second phase primary service.
$P_{2,n}^{i}(x,t) dt =$	$\label{eq:rescaled} \begin{split} & \Pr \ \{N_S(t) = n, \ x < \boldsymbol{S}_i^o\left(\boldsymbol{t}\right) \leq x + dt, \ Y(t) = 2\}, \ a \ \text{customer} \ is \ being \ \text{served} \ in \ the \ i^th \ optional \ \text{service} \ of \ the \ \text{second} \ phase, \ during \ feedback. \end{split}$
$BR_{1,n}^{0}(x, y, t) dt =$	Pr {N _s (t) = n, $S^{\circ}(t) = x, y < (R_1^0)^{\circ}(t) \le y + dt, Y(t) = 3$ }, a customer is waiting for the first primary service due to breakdown.
$BR_{1,n}(x, y, t) dt =$	Pr {N _s (t) = n, $S^{\circ}(t) = x$, $y < (R_1)^{\circ}(t) \le y + dt$, $Y(t) = 4$ }, a customer is waiting for the first phase – feedback services due to breakdown.
For $1 \leq i \leq C$; $n \geq 1$	
$BR_{2,n}^{(i,0)}(x, y, t) dt =$	Pr {N _S (t) = n, $S_i^o(t) = x, y < (R_2^{(i,0)})^o(t) \le y + dt, Y(t) = 5$ }, a customer is waiting for the second i th phase fresh service due to breakdown.
$BR^{i}_{2,n}(\mathbf{x},\mathbf{y},\mathbf{t})dt =$	Pr {N _S (t) = n, $S_i^o(t) = x, y < (R_2^i)^o(t) \le y + dt, Y(t) = 6$ }, a customer is waiting for the second i th phase service, during feedback due to breakdown.
$Q_n(x, t) dt =$	$\label{eq:relation} \begin{split} Pr \ \{N_S(t) = n, \ x < V^o \left(t \right) \leq x + dt, \ Y(t) = 7 \}, \ \text{when the server is in vacation state.} \\ (n \geq 0) \end{split}$

1.1.1.2 System Size Distribution at Random Epoch

Observing the changes of states during the interval (t, t + Δt) for any time t, the steady state equations are given by :

System in Empty State

$$\lambda PI = Q_0(0) + \sum_{i=1}^{C} \left(\mathsf{P}_{2,1}^{(i,0)}(0) + \mathsf{P}_{2,1}^{i}(0) \right) (1-f) (1-p)$$

Vacation State

$$\begin{aligned} -\frac{d}{d x} Q_n(x) &= -\lambda Q_n(x) + \lambda (1 - \delta_{0,n}) \sum_{k=1}^n Q_{n-k}(x) g_k \\ &+ \sum_{i=1}^C \left(\mathsf{P}_{2,n+1}^{(i,0)}(0) + \mathsf{P}_{2,n+1}^i(0) \right) (1 - f) p v(x) \ n \ge 0 \end{aligned}$$

Busy with First Phase – Primary Service

$$-\frac{d}{dx} P_{1,n}^{0}(x) = -(\lambda + a_{1}^{0}) P_{1,n}^{0}(x) + PI\lambda g_{n} s(x) + BR_{1,n}^{0}(x,0)$$

$$+ \sum_{i=1}^{C} \left(\mathsf{P}_{2,n+1}^{(i,0)}(0) + \mathsf{P}_{2,n+1}^{i}(0) \right) (1-f) (1-p) s(x) + Q_{n}(0) s(x)$$

+ $\lambda (1-\delta_{1,n}) \sum_{k=1}^{n-1} \mathsf{P}_{1,n-k}^{0}(x) g_{k}$

Busy with First Phase Feedback Service

$$\begin{aligned} -\frac{d}{dx} P_{1,n}(x) &= -(\lambda + a_1) P_{1,n}(x) + BR_{1,n}(x,0) + \sum_{i=1}^{C} \left(\mathsf{P}_{2,n}^{(i,0)}(0) + \mathsf{P}_{2,n}^{i}(0) \right) f_1 s(x) \\ &+ (1 - \delta_{1,n}) \lambda \sum_{k=1}^{n-1} \mathsf{P}_{1,n-k}(x) g_k \end{aligned}$$

Busy with Second Phase Primary Service

$$\begin{split} -\frac{d}{d\,x} \; P_{2,n}^{(i,0)} \left(x\right) &= -(\lambda + \, a_{2}^{(i,0)} \,) \; P_{2,n}^{(i,0)} \left(x\right) + (1 - \delta_{1,n}) \, \lambda \sum_{k=1}^{n-1} \; P_{2,n-k}^{(i,0)} \left(x\right) g_{k} \\ &+ \; P_{1,n}^{o} \left(0\right) r_{i} \; S_{i}(x) + \; BR_{2,n}^{(i,0)} \left(x,0\right), \quad 1 \leq i \leq C, \; n \geq 1 \end{split}$$

Busy with Second Phase Feedback Service $(1 \le C)$

$$\begin{aligned} -\frac{d}{d x} P_{2,n}^{i}(x) &= -(\lambda + a_{2}^{i}) P_{2,n}^{i}(x) + \lambda (1 - \delta_{1,n}) \sum_{k=1}^{n-1} P_{2,n-k}^{i}(x) g_{k} + BR_{2,n}^{i}(x, 0) \\ &+ \sum_{\ell=1}^{C} \left(P_{2,n}^{\ell}(0) + P_{2,n}^{(\ell,0)}(0) \right) f_{2}r_{i} S_{i}(x) + P_{1,n}(0) r_{i} S_{i}(x), \end{aligned}$$

Breakdown in First Phase Fresh Service

$$\begin{aligned} &-\frac{\partial}{\partial y} BR_{1,n}^{0}(x,y) &= -\lambda BR_{1,n}^{0}(x,y) + \lambda (1-\delta_{1,n}) \sum_{k=1}^{n-1} BR_{1,n-k}^{0}(x,y) g_{k} \\ &+ a_{1}^{0} r_{1}^{0}(y) P_{1,n}^{0}(x), \end{aligned}$$
 (n ≥ 1)

Breakdown in First Phase Feedback Services

$$\begin{aligned} -\frac{\partial}{\partial y} BR_{1,n}(x, y) &= -\lambda BR_{1,n}(x, y) + \lambda (1 - \delta_{1,n}) \sum_{k=1}^{n-1} BR_{1,n-k}(x, y) g_k \\ &+ a_1 r_1(y) P_{1,n}(x), \qquad n \ge 1 \end{aligned}$$

Breakdown in Second Phase Primary Service

$$-\frac{\partial}{\partial y} BR_{2,n}^{(i,0)}(x,y) = -\lambda BR_{2,n}^{(i,0)}(x,y) + (1-\delta_{1,n})\lambda \sum_{k=1}^{n-1} BR_{2,n-k}^{(i,0)}(x,y) g_k$$

International Journal of Arts Humanities and Social Sciences V1 • I4 •

Repairable Two Phase Service M^X/G/1 Queueing Models With Infinite Number Of Immediate...

$$+ a_2^{(i,0)} r_2^{(i,0)}(y) P_{2,n}^{(i,0)}(x), \qquad 1 \le i \le C$$

Breakdown in Second Phase Feedback Services

$$\begin{aligned} -\frac{\partial}{\partial y} BR_{2,n}^{i}(x,y) &= -\lambda BR_{2,n}^{i}(x,y) + (1-\delta_{1,n})\lambda \sum_{k=1}^{n-1} BR_{2,n-k}^{i}(x,y) g_{k} \\ &+ a_{2}^{i} r_{2}^{i}(y) P_{2,n}^{i}(x), \qquad 1 \le i \le C \end{aligned}$$

The LST of the Steady-State equations are given by

$$\begin{split} \lambda \, PI &= Q_0(0) + \sum_{i=1}^C \left(P_{2,1}^{(i,0)}(0) + P_{2,1}^i(0) \right) (1-f) (1-p) (1.0) \\ \theta \, Q_n^*(\theta) - Q_n(0) &= \lambda \, Q_n^*(\theta) - \lambda \, (1-\delta_{0,n}) \sum_{k=1}^n \, Q_{n-k}^*(\theta) \, g_k \\ &- \sum_{i=1}^C \left(P_{2,n+1}^{(i,0)}(0) + P_{2,n+1}^i(0) \right) (1-f) \, p \, \, V^*(\theta) \quad n \ge 0 \quad (1.1) \\ \theta \, P_{1,n}^{0*}(\theta) - P_{1,n}^0(0) &= (\lambda + a_1^0) \, P_{1,n}^{0*}(\theta) - PI\lambda g_n \, S^*(\theta) - BR_{1,n}^{0*}(\theta, 0) \\ &- \sum_{i=1}^C \left(P_{2,n+1}^{(i,0)}(0) + P_{2,n+1}^i(0) \right) (1-f) \, (1-p) \, S^*(\theta) \\ &- Q_n(0) \, \, S^*(\theta) - \lambda \, (1-\delta_{1,n}) \sum_{k=1}^{n-1} \, P_{1,n-k}^{0*}(\theta) \, g_k \quad (1.2) \end{split}$$

$$\theta P_{1,n}^{*}(\theta) - P_{1,n}(0) = (\lambda + a_{1}) P_{1,n}^{*}(\theta) - BR_{1,n}^{*}(\theta, 0) - \sum_{i=1}^{C} (P_{2,n}^{(i,0)}(0) + P_{2,n}^{i}(0)) f_{1}S^{*}(\theta)$$

- $(1 - \delta_{1,n}) \lambda \sum_{k=1}^{n-1} P_{1,n-k}^{*}(\theta) g_{k}$ (1.3)

$$\theta P_{2,n}^{(i,0)*}(\theta) - P_{2,n}^{(i,0)}(0) = (\lambda + a_2^{(i,0)}) P_{2,n}^{(i,0)*}(\theta) - (1 - \delta_{1,n}) \lambda \sum_{k=1}^{n-1} P_{2,n-k}^{(i,0)*}(\theta) g_k$$
$$- P_{1,n}^0(0) r_i S_i^*(\theta) - B R_{2,n}^{(i,0)*}(\theta, 0), \ 1 \le i \le C$$
(1.4)

$$\begin{split} \theta \, P_{2,n}^{i*}(\theta) - P_{2,n}^{i}(0) &= (\lambda + a_{2}^{i}) \, P_{2,n}^{i*}(\theta) - (1 - \delta_{1,n}) \, \lambda \sum_{k=1}^{n-1} \, P_{2,n-k}^{i*}(\theta) \, g_{k} \\ &- B R_{2,n}^{i*}(\theta, 0) - \sum_{\ell=1}^{C} \, \left(P_{2,n}^{\ell}(0) + P_{2,n}^{(\ell,0)}(0) \right) \, f_{2} r_{i} \, S_{i}^{*}(\theta) \end{split}$$

$$- P_{1,n}(0) r_{i} S_{i}^{*}(\theta) \qquad 1 \leq i \leq C \qquad (1.5)$$

$$\theta_{1} BR_{1,n}^{0**_{1}}(\theta, \theta_{1}) - BR_{1,n}^{0*}(\theta, 0) = \lambda BR_{1,n}^{0**_{1}}(\theta, \theta_{1}) - \lambda(1 - \delta_{1,n}) \sum_{k=1}^{n-1} BR_{1,n-k}^{0**_{1}}(\theta, \theta_{1}) g_{k}$$

$$- a_{1}^{0} R_{1,n}^{0*_{1}}(\theta, 1) P_{1,n}^{0*}(\theta) \qquad (1.6)$$

$$\theta_{1} BR_{1,n}^{**_{1}}(\theta, \theta_{1}) - BR_{1,n}^{*}(\theta, 0) = \lambda BR_{1,n}^{**_{1}}(\theta, \theta_{1}) - \lambda(1 - \delta_{1,n}) \sum_{k=1}^{n-1} BR_{1,n-k}^{**_{1}}(\theta, \theta_{1}) g_{k}$$

$$- a_{1} R_{1,n}^{*_{1}}(\theta, 1) P_{1,n}^{*}(\theta) \qquad (1.7)$$

$$\theta_{1} BR_{2,n}^{(i,0)**_{1}}(\theta, \theta_{1}) - BR_{2,n}^{(i,0)*}(\theta, 0) = \lambda BR_{2,n}^{(i,0)**_{1}}(\theta, \theta_{1})$$

$$- \lambda (1 - \delta_{1,n}) \sum_{k=1}^{n-1} BR_{2,n-k}^{(i,0)*_{1}}(\theta, \theta_{1}) g_{k}$$

$$- a_{2}^{(i,0)} R_{2,n}^{(i,0)*_{1}}(\theta, 1) P_{2,n}^{(i,0)*_{1}}(\theta) \qquad (1.8)$$

$$\theta_{1} BR_{2,n}^{i**_{1}}(\theta, \theta_{1}) - BR_{2,n}^{i*}(\theta, 0) = \lambda BR_{2,n}^{i**_{1}}(\theta, \theta_{1}) P_{2,n}^{(i,0)*_{1}}(\theta) \qquad (1.9)$$

1.1.3 Probability Generating Functions

The following partial PGFs are introduced to analyse the model :

$$Q^{*}(z,\theta) = \sum_{n=0}^{\infty} Q_{n}^{*}(\theta) z^{n}, \qquad Q(z,0) = \sum_{n=0}^{\infty} Q_{n}(0) z^{n}$$

$$P_{1}^{0*}(z,\theta) = \sum_{n=1}^{\infty} P_{1,n}^{0*}(\theta) z^{n}, \qquad P_{1}^{0}(z,0) = \sum_{n=1}^{\infty} P_{1,n}^{0}(0) z^{n}$$

$$P_{1}^{*}(z,\theta) = \sum_{n=1}^{\infty} P_{1,n}^{*}(\theta) z^{n}, \qquad P_{1}(z,0) = \sum_{n=1}^{\infty} P_{1,n}(0) z^{n}$$

$$P_{2}^{(i,0)*}(z,\theta) = \sum_{n=1}^{\infty} P_{2,n}^{(i,0)*}(\theta) z^{n}, \qquad P_{2}^{(i,0)}(z,0) = \sum_{n=1}^{\infty} P_{2,n}^{(i,0)}(0) z^{n}$$

$$P_2^{i*}(z,\theta) = \sum_{n=1}^{\infty} P_{2,n}^{i*}(\theta) z^n, \qquad P_2^i(z,0) = \sum_{n=1}^{\infty} P_{2,n}^i(0) z^n \ 1 \le i \le C$$

$$BR_{1}^{0**_{1}}(z,\theta,\theta_{1}) = \sum_{n=1}^{\infty} BR_{1,n}^{0**_{1}}(\theta,\theta_{1}) z^{n}, \quad BR_{1}^{0*}(z,\theta,0) = \sum_{n=1}^{\infty} BR_{1,n}^{0}(\theta,0) z^{n}$$

Repairable Two Phase Service $M^X/G/1$ Queueing Models With Infinite Number Of Immediate...

$$BR_{1}^{**_{1}}(z,\theta,\theta_{1}) = \sum_{n=1}^{\infty} BR_{1,n}^{**_{1}}(\theta,\theta_{1}) z^{n}, \quad BR_{1}^{*}(z,\theta,0) = \sum_{n=1}^{\infty} BR_{1,n}(\theta,0) z^{n}$$
$$BR_{2}^{(i,0)^{**_{1}}}(z,\theta,\theta_{1}) = \sum_{n=1}^{\infty} BR_{2,n}^{(i,0)^{**_{1}}}(\theta,\theta_{1}) z^{n}, \quad BR_{2}^{(i,0)^{*}}(z,\theta,0) = \sum_{n=1}^{\infty} BR_{2,n}^{(i,0)}(\theta,0) z^{n}$$

$$\mathsf{BR}_{2}^{i**_{1}}(z,\theta,\theta_{1}) = \sum_{n=1}^{\infty} \mathsf{BR}_{2,n}^{i**_{1}}(\theta,\theta_{1}) z^{n}, \qquad \mathsf{BR}_{2}^{i*}(z,\theta,0) = \sum_{n=1}^{\infty} \mathsf{BR}_{2,n}^{i}(\theta,0) z^{n}$$

Multiplying the corresponding equations by suitable powers of z and adding the equations, partial generating functions are derived, through some algebraic manipulations. Equation (1.1) implies

$$(\theta - w_X(z)) Q^*(z, \theta) = Q(z, 0) - \frac{(1 - f) p V^*(\theta)}{z} \sum_{i=1}^{C} (P_2^i(z, 0) + P_2^{(i,0)}(z, 0))$$

At $\theta = w_X(z)$,

$$Q(z, 0) = \frac{(1-f) p V^{*}(w_{X}(z))}{z} \sum_{i=1}^{C} (P_{2}^{i}(z, 0) + P_{2}^{(i,0)}(z, 0)), \text{ hence}$$
(1.10)

$$Q^{*}(z,\theta) = \frac{(1-f)p}{z} \frac{(V^{*}(w_{X}(z)) - V^{*}(\theta))}{(\theta - w_{X}(z))} \sum_{i=1}^{C} (P_{2}^{i}(z,0) + P_{2}^{(i,0)}(z,0)) (1.11)$$

The partial probability generating functions of the system size, when the server is in breakdown state during first stage (primary service and feedback services) services are obtained using the equations (1.6) and (1.7) given by,

$$BR_1^{0*}(z,\theta,0) = a_1^0 P_1^{0*}(z,\theta) R_1^{0*_1}(w_X(z))$$
(1.12)

$$\mathsf{BR}_{1}^{0**_{1}}(z,\theta,\theta_{1}) = a_{1}^{0} \mathsf{P}_{1}^{0*}(z,\theta) \frac{[\mathsf{R}_{1}^{0*_{1}}(\mathsf{w}_{X}(z)) - \mathsf{R}_{1}^{0*_{1}}(\theta_{1})]}{(\theta_{1} - \mathsf{w}_{X}(z))}$$
(1.12.1)

$$BR_{1}^{*}(z, \theta, 0) = a_{1}P_{1}^{*}(z, \theta) R_{1}^{*_{1}}(w_{X}(z))$$
(1.13)

$$\mathsf{BR}_{1}^{**_{1}}(z,\theta,\theta_{1}) = \mathsf{a}_{1} \mathsf{P}_{1}^{*}(z,\theta) \frac{[\mathsf{R}_{1}^{*_{1}}(\mathsf{w}_{X}(z)) - \mathsf{R}_{1}^{*_{1}}(\theta_{1})]}{(\theta_{1} - \mathsf{w}_{X}(z))}$$
(1.13.1)

The partial probability generating functions of the system size, when the server is in breakdown state during second stage (primary and feedback) services are obtained using the equations (1.8) and (1.9) and given by,

$$\mathsf{BR}_{2}^{(i,0)^{*}}(z,\theta,0) = a_{2}^{(i,0)} \mathsf{P}_{2}^{(i,0)^{*}}(z,\theta) \mathsf{R}_{2}^{(i,0)^{*}}(\mathsf{w}_{X}(z))$$
(1.14)

$$\mathsf{BR}_{2}^{(i,0)^{**_{1}}}(z,\theta,\theta_{1}) = a_{2}^{(i,0)} \mathsf{P}_{2}^{(i,0)^{*}}(z,\theta) \frac{[\mathsf{R}_{2}^{(i,0)^{*_{1}}}(\mathsf{w}_{X}(z)) - \mathsf{R}_{2}^{(i,0)^{*_{1}}}(\theta_{1})]}{(\theta_{1} - \mathsf{w}_{X}(z))}$$
(1.15)

$$BR_{2}^{i*}(z,\theta,0) = a_{2}^{i} P_{2}^{i*}(z,\theta) R_{2}^{i*_{1}}(w_{X}(z))$$
(1.16)

International Journal of Arts Humanities and Social Sciences V1 • I4 •

Repairable Two Phase Service $M^X/G/1$ Queueing Models With Infinite Number Of Immediate...

$$\mathsf{BR}_{2}^{i**_{1}}(z,\theta,\theta_{1}) = a_{2}^{i} \mathsf{P}_{2}^{i*}(z,\theta) \frac{[\mathsf{R}_{2}^{i*_{1}}(\mathsf{w}_{X}(z)) - \mathsf{R}_{2}^{i*_{1}}(\theta_{1})]}{(\theta_{1} - \mathsf{w}_{X}(z))}$$
(1.17)

Equation (1.4) gives the generating functions of the system size when the server is busy with second stage fresh service, at the service completion epoch and at arbitrary epoch :

$$P_{2}^{(i,0)}(z,0) = r_{i}P_{1}^{0}(z,0) S_{i}^{*}(h_{a_{2}(i,0)}(w_{X}(z)))$$
(1.18)

where $h_{a_2(i,0)}(w_X(z)) = w_x(z) + a_2^{(i,0)} (1 - R_2^{(i,0)*_1}(w_x(z)))$ (1.18.1)

$$P_{2}^{(i,0)*}(z,\theta) = r_{i}P_{1}^{0}(z,0) \frac{(S_{i}^{*}(h_{a_{2}}(i,0)}(w_{X}(z))) - S_{i}^{*}(\theta))}{(\theta - h_{a_{2}}(i,0)}(w_{X}(z)))}$$
(1.19)

Similarly Equation (1.5) gives the generating functions of the system size when the server is busy with second stage i^{th} feedback service, at the service completion epoch and at arbitrary epoch :

$$P_{2}^{i}(z,0) = r_{i}S_{i}^{*}(h_{a_{2}^{i}}(w_{X}(z))) [f_{2}(\sum_{\ell=1}^{C} (P_{2}^{\ell}(z,0) + P_{2}^{(\ell,0)}(z,0)) + P_{1}(z,0)] (1.20)$$

$$P_{2}^{i^{*}}(z,\theta) = \frac{r_{i}\left(S_{i}^{*}(h_{a_{2}^{i}}(w_{X}(z))) - S_{i}^{*}(\theta)\right)}{(\theta - h_{a_{2}^{i}}(w_{X}(z)))} [f_{2}(\sum_{\ell=1}^{C}(P_{2}^{\ell}(z,0) + P_{2}^{(\ell,0)}(z,0)) + P_{1}(z,0)]$$
(1.21)

where
$$h_{a_2^i}(w_X(z)) = w_X(z) + a_2^i(1 - R_2^{i*_1}(w_X(z)))$$
 (1.21.1)

The equations (1.14) to (1.21) are obtained for $1 \le i \le C$.

Adding the equations (1.18) and (1.20) over i = 1 to C,

$$\left(\sum_{\ell=1}^{C} \left(\mathsf{P}_{2}^{(\ell,0)}(z,0) + \mathsf{P}_{2}^{\ell}(z,0)\right) = \frac{\mathsf{k}_{0}(z)\,\mathsf{P}_{1}^{0}(z,0) + \mathsf{k}(z)\,\mathsf{P}_{1}(z,0)}{1 - \mathsf{f}_{2}\,\mathsf{k}(z)} \tag{1.22}\right)$$

where
$$k_0(z) = \sum_{i=1}^{C} r_i S_i^* (h_{a_2^{(i,0)}}(w_X(z)))$$
 (1.23)

and
$$k(z) = \sum_{i=1}^{C} r_i S_i^*(h_{a_2^i}(w_X(z)))$$
 (1.23.1)

The PGF corresponding to the state, when the server is busy in first stage feedback service, is obtained by using the equation (1.3) as,

$$(\theta - h_{a_1}(w_X(z))) P_1^*(z, \theta) = P_1(z, 0) - f_1 S^*(\theta) \left(\sum_{i=1}^{C} (P_2^i(z, 0) + P_2^{(i,0)}(z, 0)) (1.24)\right)$$

Using the equation (1.22) in (1.24) and simplifying, we have at $\theta = h_{a_1}(w_X(z))$,

$$P_{1}(z, 0) = \frac{f_{1} S^{*}(h_{a_{1}}(w_{X}(z))) k_{0}(z) P_{1}^{0}(z, 0)}{1 - k(z) (f_{2} + f_{1} S^{*}(h_{a_{1}}(w_{X}(z))))}$$
(1.25)

Substituting the value of $P_1(z, 0)$ in (1.24) and simplifying we get

$$P_{1}^{*}(z,\theta) = \frac{f_{1}k_{0}(z)P_{1}^{0}(z,0)}{1-k(z)(f_{2}+f_{1}S^{*}(h_{a_{1}}(w_{X}(z))))} \frac{(S^{*}(h_{a_{1}}(w_{X}(z)))-S^{*}(\theta))}{(\theta-h_{a_{1}}(w_{X}(z)))}$$
(1.26)

Using the equations (1.22) and (1.25) in (1.11),

$$Q^{*}(z,\theta) = \frac{(1-f)p}{z} \frac{k_{0}(z)P_{1}^{0}(z,0)}{[1-k(z)(f_{2}+f_{1}S^{*}(h_{a_{1}}(w_{X}(z))))]} \frac{(V^{*}(w_{X}(z)))-V^{*}(\theta))}{(\theta-w_{X}(z))}$$

(1.26.1)

Next to calculate the PGF corresponding to the state when the server is busy in first stage fresh service, the equation (1.2) is used and it is found that,

$$\begin{split} \theta P_{1}^{0^{*}}(z,\theta) &- P_{1}^{0}(z,0) \\ &= (\lambda + a_{1}^{0}) P_{1}^{0^{*}}(z,\theta) - PI\lambda X(z) S^{*}(\theta) - BR_{1}^{0^{*}}(z,\theta,0) \\ &- (1-f) (1-p) \frac{S^{*}(\theta)}{z} \sum_{i=1}^{C} \left[P_{2}^{(i,0)}(z,0) - P_{2,1}^{(i,0)}(0) z + P_{2}^{i}(z,0) - P_{2,1}^{i}(z,0) z \right] \\ &- \lambda X(z) P_{1}^{0^{*}}(z,\theta) - S^{*}(\theta) \left[Q(z,0) - Q_{0}(0) \right] \end{split}$$
(1.27)

The equations (1.12), (1.22), (1.0), (1.10) together with (1.27) give

$$(\theta - h_{a_{1}^{0}}(w_{X}(z))) P_{1}^{0^{*}}(z, \theta)$$

$$= P_{1}^{0}(z, 0) \left(\frac{\left[z[1 - k(z) (f_{2} + f_{1} S^{*}(h_{a_{1}}(w_{X}(z))))] - (g_{1} - g_{1}^{*}(\theta) (1 - f) (1 - g + g V^{*}(w_{X}(z))) K_{0}(z)) \right]}{z (1 - k(z) [f_{2} + f_{1} S^{*}(h_{a_{1}}(w_{X}(z)))])} + S^{*}(\theta) PI_{W_{X}(z)}(1.28)$$

At $\theta = h_{a_1}^0(w_X(z))$

$$P_{1}^{0}(z,0) = \frac{-w_{X}(z) \operatorname{PI} S^{*}(h_{a_{1}^{0}}(w_{X}(z))) z S_{F}(w_{X}(z))}{D_{1,IF}^{BV}(z)}$$
(1.29)

where
$$S_F^*(w_X(z)) = [1 - k(z) (f_2 + f_1 S^*(h_{a_1}(w_X(z))))]$$

and $D_{1,IF}^{BV}(z) = z S_F^*(w_X(z)) - S^*(h_{a_1^0}(w_X(z))) (1-f) (1-p+pV^*(w_X(z))) k_0(z)$ (1.30)

Substituting (1.29) in (1.28) and simplifying we get

$$P_{1}^{0^{*}}(z,\theta) = \frac{w_{X}(z) \operatorname{PI} z \, S_{F}^{*}(w_{X}(z)) \left[S^{*}(\theta) - S^{*}(h_{a_{1}^{0}}(w_{X}(z)))\right]}{D_{1,IF}^{BV}(z) \left(\theta - h_{a_{1}^{0}}(w_{X}(z))\right)} (1.31)$$

Thus the partial generating functions corresponding to different states at arbitrary epochs are calculated using the respective equations and are given by :

$$P_{1}^{0^{*}}(z,0) = \frac{z \operatorname{PI} w_{X}(z) S_{F}^{*}(w_{X}(z)) [S^{*}(h_{a_{1}^{0}}(w_{X}(z))) - 1]}{D_{1,IF}^{BV}(z) h_{a_{1}^{0}}(w_{X}(z))}$$
(1.32.1)

$$P_{1}^{*}(z,0) = \frac{z \operatorname{PI} w_{X}(z) f_{1} k_{0}(z) S^{*}(h_{a_{1}^{0}}(w_{X}(z))) (S^{*}(h_{a_{1}}(w_{X}(z))) - 1)}{D_{1,IF}^{BV}(z) h_{a_{1}}(w_{X}(z))}$$
(1.32.2)

For $1 \le i \le C$,

$$P_{2}^{(i,0)^{*}}(z,0) = \frac{z \operatorname{PI}(w_{X}(z)) r_{i} S^{*}(h_{a_{1}^{0}}(w_{X}(z))) S_{F}^{*}(w_{X}(z)) [S_{i}^{*}(h_{a_{2}^{(i,0)}}(w_{X}(z))) - 1]}{D_{1,IF}^{\mathsf{BV}}(z) h_{a_{2}^{(i,0)}}(w_{X}(z))} (1.32.3)$$

 $P_2^{i^*}(z,0)$

$$= \frac{z \operatorname{PI} w_{X}(z) r_{i} k_{0}(z) S^{*}(h_{a_{1}^{0}}(w_{X}(z))) [f_{2} + f_{1} S^{*}(h_{a_{1}}(w_{X}(z)))] [S_{i}^{*}(h_{a_{2}^{i}}(w_{X}(z))) - 1]}{D_{1,IF}^{BV}(z) h_{a_{2}^{i}}(w_{X}(z))}$$
(1.32.4)
$$Q^{*}(z,0) = \frac{\operatorname{PI} (1 - f) p k_{0}(z) S^{*}(h_{a_{1}^{0}}(w_{X}(z))) (V^{*}(w_{X}(z)) - 1)}{D_{1,IF}^{BV}(z)}$$
(1.32.5)

$$BR_{1}^{0^{**_{1}}}(z,0,0) = \frac{a_{1}^{0}P_{1}^{0^{*}}(z,0)(1-R_{1}^{0^{*_{1}}}(w_{X}(z)))}{w_{X}(z)}$$
(1.32.6)

$$BR_{1}^{**_{1}}(z,0,0) = \frac{a_{1}P_{1}^{*}(z,0)(1-R_{1}^{*_{1}}(w_{X}(z)))}{w_{X}(z)}$$
(1.32.7)

$$\mathsf{BR}_{2}^{(i,0)^{**_{1}}}(z,0,0) = \frac{\mathsf{a}_{2}^{(i,0)} \mathsf{P}_{2}^{(i,0)^{*}}(z,0) (1 - \mathsf{R}_{2}^{(i,0)^{*_{1}}}(\mathsf{w}_{X}(z)))}{\mathsf{w}_{X}(z)}$$
(1.32.8)

Repairable Two Phase Service $M^X/G/1$ Queueing Models With Infinite Number Of Immediate...

$$BR_{2}^{i^{**1}}(z,0,0) = \frac{a_{2}^{i}P_{2}^{i^{*}}(z,0)(1-R_{2}^{i^{*1}}(w_{X}(z)))}{w_{X}(z)}$$
(1.32.9)

where $D_{1,IF}^{BV}(z)$, k(z), $k_0(z)$ are given by the equations (1.30), (1.23.1) and (1.23) respectively.

To derive the total PGF of the system size distribution, the following generating functions are considered.

 $P_{Idle}(z) = Probability$ generating function of the system size when the server is idle in idle state

=
$$PI + Q^{*}(z, 0)$$

= $\frac{PI}{D_{1,IF}^{BV}(z)} [z S_{F}^{*}(w_{X}(z)) - S^{*}(h_{a_{1}^{0}}(w_{X}(z)))(1 - f)k_{0}(z)]$ (1.33)

 $P_{Comp}(z) = \text{The PGF of the system size when server is busy or in} \qquad \text{breakdown state}$ $= P_1^{0^*}(z,0) + BR_1^{0^{**_1}}(z,0,0) + P_1^*(z,0) + BR_1^{**_1}(z,0,0)$ $+ \sum_{i=1}^{C} [P_2^{(i,0)^*}(z,0) + BR_2^{(i,0)^{**_1}}(z,0,0) + P_2^{i^*}(z,0) + BR_2^{i^{**_1}}(z,0,0)]$ $\frac{PI z}{D_{1,IF}^{BV}(z)} [S^*(h_{a_1^0}(w_X(z))) k_0(z) (1-f) - S_F^*(w_X(z))] \qquad (1.34)$

Thus the total PGF of the system size distribution is given by

$$P_{IF}^{BV}(z) = P_{Idle}(z) + P_{Comp}(z)$$

$$= \frac{PI(z-1)(1-f)k_0(z)S^*(h_{a_1^0}(w_X(z)))}{D_{I,IF}^{BV}(z)}$$
(1.35)

where $D_{1,IF}^{BV}(z)$ is given by the equation (1.30). PI can be calculated by using the normalizing condition $P_{IF}^{BV}(1) = 1$ and found to be $PI = 1 - \rho_{1,IF}^{BV}$ where

(1.36)

$$\rho_{1,IF}^{BV} = \lambda E(X)[pE(V) + E(H_1^0) + \frac{f}{1-f} \sum_{i=1}^{C} r_i E(H_2^i) + \sum_{i=1}^{C} r_i E(H_2^{(i,0)}) + \frac{f_1}{1-f} E(H_1)] (1.37)$$

The measures E(H) s' are obtained from the LST of random variables,

International Journal of Arts Humanities and Social Sciences V1 • I4 •

=

and are given by :

E(H ₁)	=	$E(S) (1 + a_1 E(R_1))$	(1.37.1))
E(H ⁰ ₁)	=	$E(S) (1 + a_1^0 E(R_1^0))$	(1.37.2))
${\sf E}({\sf H}_2^{({\rm i},0)})$	=	$E(S_i) (1 + a_2^{(i,0)} E(R_2^{(i,0)}))$		(1.37.3)
$E(H_2^{i})$	=	$E(S_i) (1 + a_2^i E(R_2^i))$		(1.37.4)
$E(H_{1}^{2})$	=	$E(S) a_1 E(R_1^2) + E(S^2) (1 + a_1 E(R_1))^2$	(1.37.5))
$E((H_1^0)^2)$	=	$E(S) a_1^0 E((R_1^0)^2) + E(S^2) (1 + a_1^0 E(R_1^0))^2$	(1.37.6))
$E((H_2^{(i,0)})^2)$	=	$E(S_i) a_2^{(i,0)} E((R_2^{(i,0)})^2) + E(S_i^2)(1 + a_2^{(i,0)} E(R_2^{(i,0)}))$) ² (1.37.7	7)
$E((H_2^i)^2)$	=	$E(S_i) a_2^i E((R_2^i)^2) + E(S_i^2) (1 + a_2^i E(R_2^i))^2$	(2	1.37.8)

Hence

$$P_{IF}^{BV}(z) = \frac{(1 - \rho_{1,IF}^{BV})(z - 1)(1 - f)k_0(z)S^*(h_{a_1^0}(w_X(z)))}{D_{1,IF}^{BV}(z)}$$
(1.38)

8.1.1.4 Decomposition Property

Using equation (1.33), equation (1.38) can be re-written as

$$P_{IF}^{BV}(z) = \frac{(z-1)(1-\rho_{1,IF}) (1-f)k_0(z)S^*(h_{a_1^0}(w_X(z)))}{[zS_F^*(w_X(z))-S^*(h_{a_1^0}(w_X(z)))(1-f)k_0]} \left(\frac{P_{Idle}(z)}{P_{Idle}(1)}\right)$$
(1.39)

where

$$\rho_{1,IF} = \lambda E(X) \left[E(H_1^0) + \sum_{i=1}^C r_i E(H_2^{(i,0)}) + \frac{f}{1-f} \sum_{i=1}^C r_i E(H_2^i) + \frac{f_1}{1-f} E(H_1) \right] (1.40)$$

= $\rho_{1,IF}^{BV} - p \lambda E(X) E(V)$

and $E(H_1^0)$, $E(H_1)$, $E(H_2^i)$, $E(H_2^{(i,0)})$ are given by the equations (1.38.1) to (1.38.4).

Under the steady state condition $\rho_{1,IF}^{BV} < 1$, the PGF of the stationary system size of the queueing model under consideration is the product of the PGF of the system size of $M^X/G, G_{i(1 \le i \le C)}/1$ queueing system with infinite feedback and service interruption (without vacation) and the distribution of the conditional system size during the idle period given that the server is idle.

1.1.5 Queue Size Distribution at Departure Epoch

If
$$\pi_n^+$$
 denotes the probability that there are n customers in the system at departure epoch, then $\pi_n^+ = D_1 \left[(1-f) \sum_{i=1}^{C} P_{2,n+1}^{(i,0)}(0) + P_{2,n+1}^i(0) \right]$, with the normalizing constant D_1 .

The PGF $\pi^+(z)$ of the queue size distribution { π_n^+ ; $n \ge 0$ } at departure epoch is given by

$$\pi^{+}(z) = \sum_{n=0}^{\infty} \pi_{n}^{+} z^{n} = \frac{D_{1}}{z} (1-f) \sum_{i=1}^{C} (P_{2}^{(i,0)}(z,0) + P_{2}^{i}(z,0)) = \frac{D_{1}}{z-1} \lambda(X(z)-1) P_{IF}^{BV}(z)$$

(from (1.20) and (1.18)

Evaluating D₁ using normalizing condition, $\pi^{+}(z) = \frac{(X(z)-1)}{E(X)(z-1)} P_{IF}^{BV}(z)$

1.1.6 Performance Measures

(i) The probability that the server is on vacation state (P_v) is

$$P_{v} = \lim_{z \to 1} Q^{*}(z, 0) = p\lambda E(X) E(V)$$

(ii) The probability that the server is busy is

$$P_{\text{busy}} = P_1^0 + P_1 + \sum_{i=1}^{C} (P_2^{(i,0)} + P_2^i)$$

=
$$\lim_{z \to 1} \left[P_1^{0^*}(z,0) + P_1^*(z,0) + \sum_{i=1}^{C} P_2^{(i,0)^*}(z,0) + \sum_{i=1}^{C} P_2^{i^*}(z,0) \right]$$

=
$$\frac{\lambda E(X)}{1 - f} \left[E(S)(1 - f_2) + \sum_{i=1}^{C} r_i E(S_i) \right]$$

(iii) The probability that the server is in breakdown state (P_{br}) is obtained by,

$$P_{br} = \lim_{z \to 1} \left[P_{br_1^0} + P_{br_1} + \sum_{i=1}^{C} \left(P_{br_2^{(i,0)}} + P_{br_2^i} \right) \right]$$

= $\lambda E(X) \left\{ E(S) \left[a_1^0 E(R_1^0) + \frac{f_1}{1 - f} a_1 E(R_1) \right] + \sum_{i=1}^{C} r_i E(S_i) \left[a_2^{(i,0)} E(R_2^{(i,0)}) + \frac{f}{1 - f} a_2^i E(R_2^i) \right] \right\}$

(iv) The expected system size for the model is given by

$$L_{IF}^{BV} = \left[\frac{d}{dz} \left(P_{IF}^{BV}(z)\right]_{z=1} = \lambda E(x) \left[E(H_{1}^{0}) + \sum_{i=1}^{C} r_{i} E(H_{2}^{(i,0)})\right] + \frac{\left(-D_{1,IF}^{BV}\right)^{''}(1)}{2\left(1-f\right)\left(1-\rho_{1,IF}^{BV}\right)} (1.41)$$

where $\rho_{\text{1,IF}}^{\text{BV}}$ is given by the equation (1.37) and

$$= \lambda E(X(X-1) [f \sum_{i=1}^{C} r_i E(H_2^i) + f_1 E(H_1) + (1-f) [E(H_1^0) + pE(V) + \sum_{i=1}^{C} r_i E(H_2^{(i,0)})]] + (\lambda E(X))^2 (f \sum_{i=1}^{C} r_i E(H_2^i)^2) + f_1 E(H_1^2) + (1-f) [E((H_1^0)^2) + pE(V^2) + \sum_{i=1}^{C} r_i E((H_2^{(i,0)})^2)] + 2[\sum_{i=1}^{C} r_i E(H_2^i) f_1 E(H_1) + (1-f) [pE(V)E(H_1^0) + (pE(V) + E(H_1^0)) \sum_{i=1}^{C} r_i E((H_2^{(i,0)}))]] + 2\lambda E(X) [f \sum_{i=1}^{C} r_i E(H_2^i) + f_1 E(H_1)]$$

where $E(H_1)$, $E(H_1^0)$, $E(H_2^i)$, $E(H_2^{(i,0)})$, $E(H_1^2)$, $E((H_1^0)^2)$, $E((H_2^i)^2)$, $E((H_2^{(i,0)})^2)$ are given by the equations (1.37.1) to (1.37.8).

1.2 PARTICULAR CASES

The model of the present section considers the case in which, the customers when discontented with their services may either demand re-service from phase 1 followed by phase 2 with probability (f_1) (or) demand re-service of phase 2 type alone with probability f_2 (or) leave the system with probability $1 - (f_1 + f_2) = 1 - f$, without demanding re-services.

Case 1:

If $f_1 = 0$ (i.e., $f_2 = f$), then the PGF of the system size of the model, in which the feedback customers demand re-services only of phase 2 type is obtained from equation (1.38) and given by

$$P_{1,IF}^{BV}(z) = \frac{(1 - \rho_{1,IF}^{BV})(z - 1((1 - f) S^{*}(h_{a_{1}^{0}}(w_{X}(z)))\frac{k_{0}(z)}{1 - f k(z)}}{z - S^{*}(h_{a_{1}^{0}}(w_{X}(z)))(1 - f)(1 - p + p V^{*}(w_{X}(z)))\frac{k_{0}(z)}{1 - f k(z)}}$$
(1.42.1)

where,

$$\rho_{1,\text{IF}}^{\text{BV}} = \lambda E(X) \left(\frac{f}{1-f} \sum_{i=1}^{C} r_i E(H_2^i) + \sum_{i=1}^{C} r_i E(H_2^{i,0}) + E(H_1^0) + p E(V) \right) (1.42.2)$$

Case 2 :

If $f_2 = 0$ then the PGF of the system size of the model in which all the feedback customers repeat services from phase 1 followed by phase 2 is obtained from equation (1.38) and given by

International Journal of Arts Humanities and Social Sciences V1 • I4 •

 $P_{1,IF}^{BV}(z)$

$$=\frac{(1-\rho_{1,IF}^{BV})(z-1)(1-f)k_{0}(z)S^{*}(h_{a_{1}^{0}}(w_{X}(z)))}{z(1-k(z)+fS^{*}(h_{a_{1}}(w_{X}(z))))-(1-f)S^{*}(h_{a_{1}^{0}}(w_{X}(z)))k_{0}(z)(1-p+pV^{*}(w_{X}(z)))}$$
(1.43.1)

$$\rho_{1,\text{IF}}^{\text{BV}} = \lambda E(X) \left(\frac{f}{1-f} \sum_{i=1}^{C} r_i E(H_2^i) + \sum_{i=1}^{C} r_i E(H_2^{i,0}) + \frac{f}{1-f} E(H_1) + E(H_1^0) + p E(V) \right)$$

If we assume that, the probability with which the server takes vacation after the completion of each service (p) is zero, the arrivals follow simple Poisson process (E(X) = 0) and the server never breaks down ($a_i = 0$), then equations (1.23) and (1.23.1) give :

$$k_0(z) = k(z); S^*(h_{a_1^0}(w_X(z))) = S^*(h_a(w_X(z))) = S^*(\lambda(1-z))$$

and P(z) =
$$\frac{(1-\rho)(z-1)(1-f)A(z)}{z(1-fA(z)) - (1-f)A(z)}$$
(1.44)

 $\frac{\lambda}{1-f} (E(S) + \sum_{i=1}^{C} r_i E(S_i)) \text{ and } A(z) = k(z) S^* (\lambda (1-z))$

it is verified that under the condition $m \rightarrow \infty$, (1.44) coincides with the PGF of the system size of the two phase service reliable M/G/1 queueing model (with finite number of immediate feedback) analysed by Kalidass and Kasturi (2013).

NUMERICAL ANALYSIS

In this section numerical results are obtained to study the effects of (i) the probability that the server chooses feedback service from first phase (f_1) or from second phase (f_2) , (ii) the breakdown rates, (iii) mean repair time on the expected system size for the present model. The different distributions assumed are presented in the following table.

Random variables (Y)		Distribution F(Y)	Mean E(Y) Second orde moments E(Y)	
FPS (S)		Erlang-3 type	$\frac{1}{4}$	<u>1</u> 12
SPS (S ₁ , S ₂ , S ₃)		(Deterministic, Exponential, Gamma-3)	$(\frac{1}{3}, \frac{1}{6}, \frac{3}{5})$	$(\frac{1}{3^2}, \frac{2}{6^2}, \frac{12}{5^2})$
Vacation (V)		Gamma-2 type	2 5	<u>6</u> 25
Repair time in first phase	Primary (R ₁ ⁰)	Erlang-2 type	<u>1</u> 4	$\frac{3}{32}$
	Feedback (R ₁)	Exponential	<u>1</u> 4	<u>1</u> 8

36

Repair time in second phase, i = 1 to 3	Primary $(R_2^{(i,0)})$	Exponential	$\frac{1}{4}$	<u>1</u> 8
	Feedback (R_2^i)	Exponential	2	8
Batch size (X)		Geometric (Geo(p ₁))	$\frac{1}{1-p_1}$, $p_1 = 0.3$	$\frac{p_1 + 1}{(1 - p_1)^2}$

The parametric values (other than in the table) chosen to construct the Tables 1.1(a) and 1.1(b) are listed below.

 $(a_1^0, a_1, r_1, r_2, r_3, p, c, \lambda) = (3, 0.03, 0.5, 0.4, 0.1, 0.6, 3, 0.1)$ and

 $(a_{2,1}^0, a_{2,1}^1, a_{2,2}^0, a_{2,2}^1, a_{2,3}^0, a_{2,3}^1) = (2, 3, 5, 3, 3, 3)$. The change of parameter values are given in the corresponding tables.

Table 1.1(a) shows that the mean system size (L) for the infinite feedback queueing model increases with f_1 as well as f_2 . Table 1.1(b) shows that L increase with breakdown rate (a_1^0) and mean repair time $E(R_1^0)$. The graphical representations of 1.1(a) and 1.1(b) are given in Figures 1.1(a) and 1.1(b) respectively.

Table 1.1(a) TheMean System Length L Vs Feedback Probabilities f_1 and f_2 with p_1 = 0.4

f ₂	0.5	0.55	0.58	0.6
0.05	2.663	4.082	5.80	7.94
0.06	2.882	4.563	6.76	9.75
0.07	3.134	5.157	8.06	12.56
0.08	3.426	5.911	9.93	17.52
0.09	3.771	6.898	12.84	28.54
0.1	4.183	8.246	18.04	74.56

Table 1.1(b)Mean System Length L Vs. Breakdown Rate (a_1^0) for Different Values of Repair time E(R_1^0) with $f_1 = 0.01$, $f_2 = 0.6$, $\lambda = 0.17$ and p = 0.03

E(R ⁰ ₁) a ⁰ ₁	1/20	1/15	1/10	1/5
1	22.441	22.733	23.341	25.374
2	23.341	23.981	25.371	30.683
3	24.314	25.370	27.777	38.724
4	25.369	26.926	30.676	52.337
5	26.519	28.680	34.235	80.391
6	27.775	30.673	38.710	171.863







II. CONCLUSION

The present chapter examines $M^X/G/1$ queueing system with immediate feedbacksand multi second optional service facilities. The server operates single service in the first phase and different kinds of heterogeneous services in the second phase. A customer is said to complete the first round service if he undergoes the first phase service and any one of the second phase services. After having completed the first round service, the customer is permitted to repeat services from the second multi-optional services which may be different from the one chosen earlier. Kalidass and Kasturi (2013) analyzed a reliable Poisson arrival M/G/1 queue with two phases of heterogeneous service and a finite number of immediate Bernoulli feedbacks before leaving the system. In the present work, the author studies the M^X/G/1 queueing system where customers can feedback infinitely many times, the server undergoes unpredictable breakdowns and takes optional vacations between services. When the server fails, the customer in service may resume the service.

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