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Again About Calculating Of Limits Of Some Sequences With The Help Of Integrals

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Abstract: According to the curricular documents, learning Mathematics in school aims to raise awareness of the nature of Mathematics as a problem-solving activity, based on a body of knowledge and procedures, but also as a dynamic discipline, closely linked to society through its relevance in everyday life and its role. in Natural Sciences, in Technologies and in Social Sciences. The major meaning of the current references in teaching - learning Mathematics is shifting the emphasis from teaching knowledge that has essentially remained the same from the old programs, on capacity building, where "*using different strategies in solving problems*" occupies a special place. This paper falls into this context; continuing the steps begun some time ago, a way to calculate the limits for four categories of strings is presented here, with the particularities and related examples.
Keywords: string, convergence, limit, function, integral.

I. Putting the problem

In a recent paper (Vălcan, 2016), titled very suggestively *From Mathematics Spectacle. Examples of Good Practice for Pupils, Students and Teachers*, we have proposed, obtaining remarkable limits of sequences, at the upper level of the high school, so only using mathematical knowledge learned in school, but in a comprehensive approach. So, first we used Moivre's formula and Newton's binomial in solving three higher-level algebraic equations, after which, using Viête's relations, we got some inequalities that helped us calculate the limits of remarkable sequences. Finally, we applied the results obtained when calculating the limits of less known sequences.

In the paper *From limits of sequences to definite integrals* - (Vălcan, 2017) we have approached from the inductive but also deductive perspective the problem of calculating the limits of some special classes of sequences of real numbers, using the define integral, and in the paper *From integrable functions to limits of sequences* - (Vălcan, (I), 2018), I started the reverse: from certain definite integrals we have determined two classes of convergent sequences and their limits.

Finally, in the paper (Vălcan, (II), 2018) - entitled *The spectacles of Mathematics ... continues*, we continued the ideas presented in the papers mentioned above and we calculated the limits of other categories of sequences of real numbers.

In this paper, as mentioned above, we will continue this approach and present ways to calculate the limits of other sequences of real numbers.

II. Main results, particular cases and examples

This paper is a continuation of the paper (Vălcan, 2017) and, implicitly, of the paper (Vălcan, 2016), where I have shown that the following equalities hold:

A) $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{(q\cdot k)\cdot(q\cdot k+p)}=\frac{1}{p}\cdot\int_0^1\left[\frac{(1-x^p)\cdot x^{q-1}}{1-x^q}\right]\cdot dx$, with p, q∈**N** and p<q.

B) $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{(q\cdot k+p)\cdot(q\cdot k+r)}=\frac{1}{r-p}\cdot\int_0^1\left[\frac{(1-x^{r-p})\cdot x^{p-1}}{1-x^q}\right]\cdot dx$, with p, q, r∈**N** and p<r<q.

C) $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{(q\cdot k)\cdot(q\cdot k+p)\cdot(q\cdot k+r)}=\frac{1}{p\cdot r\cdot(r-p)}\cdot\int_0^1\left[\frac{[(r-p)-r\cdot x^p+p\cdot x^r]\cdot x^{q-1}}{1-x^q}\right]\cdot dx$,
with p, q, r∈**N** and p<r<q.

D) $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{(q\cdot k+p)\cdot(q\cdot k+r)\cdot(q\cdot k+s)}=\frac{1}{(r-p)\cdot(s-p)\cdot(s-r)}\cdot$

$$\int_0^1 \left[\frac{[(s-r)-(s-p)\cdot x^{r-p}+(r-p)\cdot x^{s-p}]\cdot x^{p-1}}{1-x^q} \right]\cdot dx$$, with p, q, r, s∈**N*** and p<r<s<q.

Also in (Vălcan, 2017) we calculated or proposed for calculation 16 limits for type A sequences, 24 limits for type B sequences, 24 limits for type C sequences and 16 limits for type D sequences.

Therefore, in the following, we will respect the ideas and numbering made in (Vălcan, 2016) and (Vălcan, 2017).

We start presenting the new results:

E) ***Theorem E***: *If p, q, r, s*∈***N**** *and p<r<s, then*:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(q\cdot k)\cdot(q\cdot k+p)\cdot(q\cdot k+r)\cdot(q\cdot k+s)}=\alpha\cdot\lim_{n\to\infty}\int_0^1\left[\frac{E(x)}{1-x^q}\right]\cdot dx, \quad \text{(E)}$$

where:

$E(x)=(\alpha_1-\alpha_2\cdot x^p+\alpha_3\cdot x^r-\alpha_4\cdot x^s)\cdot x^{q-1}$,

$$\alpha=\frac{1}{p\cdot r\cdot s\cdot(r-p)\cdot(s-p)\cdot(s-r)},$$

$\alpha_1=(r-p)\cdot(s-p)\cdot(s-r)$, $\alpha_2=r\cdot s\cdot(s-r)$, $\alpha_3=p\cdot s\cdot(s-p)$, $\alpha_4=p\cdot r\cdot(r-p)$.

Proof: For every k∈**N***,

$$\frac{1}{q\cdot k\cdot(q\cdot k+p)\cdot(q\cdot k+r)\cdot(q\cdot k+s)}=$$

$$=\alpha\cdot\left[\frac{(r-p)\cdot(s-p)\cdot(s-r)}{q\cdot k}-\frac{r\cdot s\cdot(s-r)}{q\cdot k+p}+\frac{s\cdot p\cdot(s-p)}{q\cdot k+r}-\frac{r\cdot p\cdot(r-p)}{q\cdot k+s}\right]$$

$$=\alpha\cdot\int_0^1(\alpha_1\cdot x^{q\cdot k-1}-\alpha_2\cdot x^{q\cdot k+p-1}+\alpha_3\cdot x^{q\cdot k+r-1}-\alpha_4\cdot x^{q\cdot k+s-1})\cdot dx$$

$$=\alpha\cdot\int_0^1(\alpha_1-\alpha_2\cdot x^p+\alpha_3\cdot x^r-\alpha_4\cdot x^s)\cdot x^{q\cdot k-1}\cdot dx,$$

where:

$$\alpha=\frac{1}{p\cdot r\cdot s\cdot(r-p)\cdot(s-p)\cdot(s-r)},$$

$\alpha_1=(r-p)\cdot(s-p)\cdot(s-r)$, $\alpha_2=r\cdot s\cdot(s-r)$, $\alpha_3=p\cdot s\cdot(s-p)$, $\alpha_4=p\cdot r\cdot(r-p)$.

So,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(q\cdot k)\cdot(q\cdot k+p)\cdot(q\cdot k+r)\cdot(q\cdot k+s)}=$$

$$=\alpha\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1[(\alpha_1-\alpha_2\cdot x^p+\alpha_3\cdot x^r-\alpha_4\cdot x^s)\cdot x^{q\cdot k-1}]\cdot dx$$

$$=\alpha\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}[(\alpha_1-\alpha_2\cdot x^p+\alpha_3\cdot x^r-\alpha_4\cdot x^s)\cdot x^{q\cdot k-1}]\cdot dx$$

$$=\alpha\cdot\lim_{n\to\infty}\int_0^1\left[(\alpha_1-\alpha_2\cdot x^p+\alpha_3\cdot x^r-\alpha_4\cdot x^s)\cdot\sum_{k=1}^{n}x^{q\cdot k-1}\right]\cdot dx$$

$$=\alpha\cdot\lim_{n\to\infty}\int_0^1\left[(\alpha_1-\alpha_2\cdot x^p+\alpha_3\cdot x^r-\alpha_4\cdot x^s)\cdot x^{q-1}\cdot\frac{1-x^{n\cdot q}}{1-x^q}\right]\cdot dx$$

$$=\alpha\cdot\lim_{n\to\infty}\int_0^1\left[\frac{(\alpha_1-\alpha_2\cdot x^p+\alpha_3\cdot x^r-\alpha_4\cdot x^s)\cdot x^{q-1}}{1-x^q}\right]\cdot dx$$

$$=\alpha\cdot\lim_{n\to\infty}\int_0^1\left[\frac{(\alpha_1-\alpha_2\cdot x^p+\alpha_3\cdot x^r-\alpha_4\cdot x^s)\cdot x^{(n+1)\cdot q-1}}{1-x^q}\right]\cdot dx$$

$$=\alpha\cdot\int_0^1\left[\frac{(\alpha_1-\alpha_2\cdot x^p+\alpha_3\cdot x^r-\alpha_4\cdot x^s)\cdot x^{q-1}}{1-x^q}\right]\cdot dx\ ,$$

because, according to (Vălcan, 2016, Proposition),

$$\lim_{n\to\infty}\int_0^1\left[\frac{(\alpha_1-\alpha_2\cdot x^p+\alpha_3\cdot x^r-\alpha_4\cdot x^s)\cdot x^{(n+1)\cdot q-1}}{1-x^q}\right]\cdot dx=0.$$

The following remark is required here:

Remark E0: *A simple calculation shows us that*:

$\alpha_1-\alpha_2+\alpha_3-\alpha_4=0$,

which tells us that none of the above integrals is not improper.

Corollary E1: *If* $q\in\mathbf{N}^*$, *then*:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(q\cdot k)\cdot(q\cdot k+1)\cdot(q\cdot k+2)\cdot(q\cdot k+3)}=\frac{1}{6}\cdot\int_0^1\left[\frac{x^{q-1}\cdot(1-x)^3}{1-x^q}\right]\cdot dx\ . \qquad \text{(E1)}$$

Proof: Equality (E1) is obtained from Theorem (E), for:

p=1, r=2 and s=3,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{q\cdot k\cdot(q\cdot k+1)\cdot(q\cdot k+2)\cdot(q\cdot k+3)}=\frac{1}{6}\cdot\left(\frac{1}{q\cdot k}-\frac{3}{q\cdot k+1}+\frac{3}{q\cdot k+2}-\frac{1}{q\cdot k+3}\right)$$

$$=\frac{1}{6}\cdot\int_0^1(x^{q\cdot k-1}-3\cdot x^{q\cdot k}+3\cdot x^{q\cdot k+1}-x^{q\cdot k+2})\cdot dx=\frac{1}{6}\cdot\int_0^1[(1-3\cdot x+3\cdot x^2-x^3)\cdot x^{q\cdot k-1}]\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1[(1-x)^3\cdot x^{q\cdot k-1}]\cdot dx\ .$$

So,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(q\cdot k)\cdot(q\cdot k+1)\cdot(q\cdot k+2)\cdot(q\cdot k+3)}=\frac{1}{6}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1[(1-x)^3\cdot x^{q\cdot k-1}]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}[(1-x)^3\cdot x^{q\cdot k-1}]\cdot dx=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot\sum_{k=1}^{n}x^{q\cdot k-1}\right]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot x^{q-1}\cdot\frac{1-x^{n\cdot q}}{1-x^q}\right]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[\frac{x^{q-1}\cdot(1-x)^3}{1-x^q}\right]\cdot dx-\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[\frac{x^{(n+1)\cdot q-1}\cdot(1-x)^3}{1-x^q}\right]\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1\left[\frac{x^{q-1}\cdot(1-x)^3}{1-x^q}\right]\cdot dx\ ,$$

because, according to (Vălcan, 2016, Proposition),

$$\lim_{n\to\infty}\int_0^1\left[\frac{x^{(n+1)\cdot q-1}\cdot(1-x)^3}{1-x^q}\right]\cdot dx=0.$$

Example E1.1: The following equality holds:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k\cdot(k+1)\cdot(k+2)\cdot(k+3)}=\frac{1}{18}\ . \qquad \text{(E1.1)}$$

Proof: Equality (E1.1) is obtained from equality (E1), for:

q=1,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{k\cdot(k+1)\cdot(k+2)\cdot(k+3)}=\frac{1}{6}\cdot\left(\frac{1}{k}-\frac{3}{k+1}+\frac{3}{k+2}-\frac{1}{k+3}\right)$$

$$=\frac{1}{6}\cdot\int_0^1(x^{k-1}-3\cdot x^k+3\cdot x^{k+1}-x^{k+2})\cdot dx=\frac{1}{6}\cdot\int_0^1[(1-3\cdot x+3\cdot x^2-x^3)\cdot x^{k-1}]\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1[(1-x)^3\cdot x^{k-1}]\cdot dx\,.$$

So,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k\cdot(k+1)\cdot(k+2)\cdot(k+3)}=\frac{1}{6}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1[(1-x)^3\cdot x^{k-1}]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}[(1-x)^3\cdot x^{k-1}]\cdot dx=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot\sum_{k=1}^{n}x^{k-1}\right]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot\frac{1-x^n}{1-x}\right]\cdot dx=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1[(1-x)^2\cdot(1-x^n)]\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1(1-x)^2\cdot dx-\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1[(1-x)^2\cdot x^n]\cdot dx=\frac{1}{6}\cdot\int_0^1(1-2\cdot x+x^2)\cdot dx$$

$$=\frac{1}{6}\cdot\left(x-2\cdot\frac{x^2}{2}+\frac{x^3}{3}\right)\Bigg|_0^1=\frac{1}{6}\cdot\left(1-2\cdot\frac{1}{2}+\frac{1}{3}\right)=\frac{1}{18}\,,$$

since:

$$\lim_{n\to\infty}\int_0^1[(1-x)^2\cdot x^n]\cdot dx=0.$$

Example E1.2: The following equality holds:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k+1)\cdot(2\cdot k+2)\cdot(2\cdot k+3)}=\frac{1}{6}\cdot\left(\frac{17}{6}-4\cdot\ln 2\right).\qquad\text{(E1.2)}$$

Proof: Equality (E1.2) is obtained from equality (E1), for:

q=2,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{2\cdot k\cdot(2\cdot k+1)\cdot(2\cdot k+2)\cdot(2\cdot k+3)}=\frac{1}{6}\cdot\left(\frac{1}{2\cdot k}-\frac{3}{2\cdot k+1}+\frac{3}{2\cdot k+2}-\frac{1}{2\cdot k+3}\right)$$

$$=\frac{1}{6}\cdot\int_0^1(x^{2\cdot k-1}-3\cdot x^{2\cdot k}+3\cdot x^{2\cdot k+1}-x^{2\cdot k+2})\cdot dx=\frac{1}{6}\cdot\int_0^1[(1-3\cdot x+3\cdot x^2-x^3)\cdot x^{2\cdot k-1}]\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1[(1-x)^3\cdot x^{2\cdot k-1}]\cdot dx\,.$$

So,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k+1)\cdot(2\cdot k+2)\cdot(2\cdot k+3)}=\frac{1}{6}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1[(1-x)^3\cdot x^{2\cdot k-1}]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}[(1-x)^3\cdot x^{2\cdot k-1}]\cdot dx=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot\sum_{k=1}^{n}x^{2\cdot k-1}\right]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot x\cdot\frac{1-x^{2\cdot n}}{1-x^2}\right]\cdot dx=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot x\cdot(1-x^n)}{1+x}\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1\frac{(1-x)^2\cdot x}{1+x}\cdot dx\text{ - }\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot x^{n+1}}{1+x}\cdot dx=\frac{1}{6}\cdot\int_0^1\left(x^2-3\cdot x+4-\frac{4}{x+1}\right)\cdot dx$$

$$=\frac{1}{6}\cdot\left(4\cdot x-3\cdot\frac{x^2}{2}+\frac{x^3}{3}-4\cdot\ln(x+1)\right)\Bigg|_0^1=\frac{1}{6}\cdot\left(\frac{17}{6}-4\cdot\ln 2\right),$$

since,

$$\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot x^{n+1}}{1+x}\cdot dx=0.$$

Example E1.3: The following equality holds:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k+1)\cdot(3\cdot k+2)\cdot(3\cdot k+3)}=\frac{1}{18}\cdot\left(\frac{11}{2}-\sqrt{3}\cdot\pi\right).\qquad\text{(E1.3)}$$

Proof: Equality (E1.3) is obtained from equality (E1), for:

q=3,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{3\cdot k\cdot(3\cdot k+1)\cdot(3\cdot k+2)\cdot(3\cdot k+3)}=\frac{1}{6}\cdot\left(\frac{1}{3\cdot k}-\frac{3}{3\cdot k+1}+\frac{3}{3\cdot k+2}-\frac{1}{3\cdot k+3}\right)$$

$$=\frac{1}{6}\cdot\int_0^1(x^{3\cdot k-1}-3\cdot x^{3\cdot k}+3\cdot x^{3\cdot k+1}-x^{3\cdot k+2})\cdot dx=\frac{1}{6}\cdot\int_0^1[(1-3\cdot x+3\cdot x^2-x^3)\cdot x^{3\cdot k-1}]\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1[(1-x)^3\cdot x^{3\cdot k-1}]\cdot dx.$$

So,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k+1)\cdot(3\cdot k+2)\cdot(3\cdot k+3)}=\frac{1}{6}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1[(1-x)^3\cdot x^{3\cdot k-1}]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}[(1-x)^3\cdot x^{3\cdot k-1}]\cdot dx=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot\sum_{k=1}^{n}x^{3\cdot k-1}\right]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot x^2\cdot\frac{1-x^{3\cdot n}}{1-x^3}\right]\cdot dx=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot x^2\cdot(1-x^{3\cdot n})}{1+x+x^2}\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1\frac{(1-x)^2\cdot x^2}{1+x+x^2}\cdot dx\text{ - }\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot x^{3\cdot n+2}}{1+x+x^2}\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1\left(x^2-3\cdot x+3-\frac{3}{x^2+x+1}\right)\cdot dx=\frac{1}{6}\cdot\left(3\cdot x-3\cdot\frac{x^2}{2}+\frac{x^3}{3}-2\cdot\sqrt{3}\cdot\operatorname{arctg}\left(\frac{2\cdot x+1}{\sqrt{3}}\right)\right)\Bigg|_0^1$$

$$=\frac{1}{6}\cdot\left(\frac{11}{6}-\frac{\sqrt{3}\cdot\pi}{3}\right),$$

since,

$$\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot x^{3\cdot n+2}}{1+x+x^2}\cdot dx=0.$$

Example E1.4: The following equality holds:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(4\cdot k)\cdot(4\cdot k+1)\cdot(4\cdot k+2)\cdot(4\cdot k+3)}=\frac{1}{12}\cdot\left(\frac{11}{3}-3\cdot\ln 2-\frac{\pi}{2}\right).\qquad\text{(E1.4)}$$

Proof: Equality (E1.4) is obtained from equality (E1), for:

q=4,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{4\cdot k\cdot(4\cdot k+1)\cdot(4\cdot k+2)\cdot(4\cdot k+3)}=\frac{1}{6}\cdot\left(\frac{1}{4\cdot k}-\frac{3}{4\cdot k+1}+\frac{3}{4\cdot k+2}-\frac{1}{4\cdot k+3}\right)$$

$$=\frac{1}{6}\cdot\int_0^1(x^{4\cdot k-1}-3\cdot x^{4k}+3\cdot x^{4\cdot k+1}-x^{4\cdot k+2})\cdot dx=\frac{1}{6}\cdot\int_0^1[(1-3\cdot x+3\cdot x^2-x^3)\cdot x^{4\cdot k-1}]\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1[(1-x)^3\cdot x^{4\cdot k-1}]\cdot dx\,.$$

So,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(4\cdot k)\cdot(4\cdot k+1)\cdot(4\cdot k+2)\cdot(4\cdot k+3)}=\frac{1}{6}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1[(1-x)^3\cdot x^{4\cdot k-1}]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}[(1-x)^3\cdot x^{4\cdot k-1}]\cdot dx=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot\sum_{k=1}^{n}x^{4\cdot k-1}\right]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot x^3\cdot\frac{1-x^{4\cdot n}}{1-x^4}\right]\cdot dx=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot x^3\cdot(1-x^{4\cdot n})}{1+x+x^2+x^3}\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1\frac{(1-x)^2\cdot x^3}{1+x+x^2+x^3}\cdot dx-\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot x^{4\cdot n+3}}{1+x+x^2+x^3}\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1\left(x^2-3\cdot x+3-2\cdot\frac{1}{x+1}+\frac{1}{2}\cdot\frac{2\cdot x}{x^2+1}-\frac{1}{x^2+1}\right)\cdot dx$$

$$=\frac{1}{6}\cdot\left(3\cdot x-3\cdot\frac{x^2}{2}+\frac{x^3}{3}-2\cdot\ln(x+1)+\frac{1}{2}\cdot\ln(x^2+1)-\operatorname{arctg}x\right)\Bigg|_0^1$$

$$=\frac{1}{6}\cdot\left(3-\frac{3}{2}+\frac{1}{3}-2\cdot\ln 2+\frac{1}{2}\cdot\ln 2-\frac{\pi}{4}\right)=\frac{1}{12}\cdot\left(\frac{11}{3}-3\cdot\ln 2-\frac{\pi}{2}\right)$$

since,

$$\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot x^{4\cdot n+3}}{1+x+x^2+x^3}\cdot dx=0.$$

Corollary E2: *If $q\in\mathbf{N}^*$, then*:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(q\cdot k)\cdot(q\cdot k+1)\cdot(q\cdot k+2)\cdot(q\cdot k+4)}=\frac{1}{24}\cdot\int_0^1\left[\frac{x^{q-1}\cdot(1-x)^3\cdot(x+3)}{1-x^q}\right]\cdot dx\,. \qquad \text{(E2)}$$

***Proof*:** Equality (E2) is obtained from Theorem (E), for:

p=1, r=2 and s=4,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{q\cdot k\cdot(q\cdot k+1)\cdot(q\cdot k+2)\cdot(q\cdot k+4)}=\frac{1}{24}\cdot\left(\frac{3}{q\cdot k}-\frac{8}{q\cdot k+1}+\frac{6}{q\cdot k+2}-\frac{1}{q\cdot k+4}\right)$$

$$=\frac{1}{24}\cdot\int_0^1(3\cdot x^{q\cdot k-1}-8\cdot x^{q\cdot k}+6\cdot x^{q\cdot k+1}-x^{q\cdot k+3})\cdot dx=\frac{1}{24}\cdot\int_0^1[(3-8\cdot x+6\cdot x^2-x^4)\cdot x^{q\cdot k-1}]\cdot dx\,.$$

So,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(q\cdot k)\cdot(q\cdot k+1)\cdot(q\cdot k+2)\cdot(q\cdot k+4)}=\frac{1}{24}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1[(3-8\cdot x+6\cdot x^2-x^4)\cdot x^{q\cdot k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}[(3-8\cdot x+6\cdot x^2-x^4)\cdot x^{q\cdot k-1}]\cdot dx=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(3-8\cdot x+6\cdot x^2-x^4)\cdot\sum_{k=1}^{n}x^{q\cdot k-1}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(3-8\cdot x+6\cdot x^2-x^4)\cdot x^{q-1}\cdot\frac{1-x^{n\cdot q}}{1-x^q}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[\frac{x^{q-1}\cdot(3-8\cdot x+6\cdot x^2-x^4)}{1-x^q}\right]\cdot dx$$

$$-\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[\frac{x^{(n+1)\cdot q-1}\cdot(3-8\cdot x+6\cdot x^2-x^4)}{1-x^q}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\left[\frac{x^{q-1}\cdot(3-8\cdot x+6\cdot x^2-x^4)}{1-x^q}\right]\cdot dx=\frac{1}{24}\cdot\int_0^1\left[\frac{x^{q-1}\cdot(1-x)^3\cdot(x+3)}{1-x^q}\right]\cdot dx\,,$$

because, according to (Vălcan, 2016, Proposition),

$$\lim_{n\to\infty}\int_0^1\left[\frac{x^{(n+1)\cdot q-1}\cdot(3-8\cdot x+6\cdot x^2-x^4)}{1-x^q}\right]\cdot dx=0.$$

Example E2.1: The following equality holds:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k\cdot(k+1)\cdot(k+2)\cdot(k+4)}=\frac{13}{288}. \tag{E2.1}$$

Proof: Equality (E2.1) is obtained from equality (E2), for:

q=1,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{k\cdot(k+1)\cdot(k+2)\cdot(k+4)}=\frac{1}{24}\cdot\left(\frac{3}{k}-\frac{8}{k+1}+\frac{6}{k+2}-\frac{1}{k+4}\right)$$

$$=\frac{1}{24}\cdot\int_0^1(3\cdot x^{k-1}-8\cdot x^k+6\cdot x^{k+1}-x^{k+3})\cdot dx=\frac{1}{24}\cdot\int_0^1[(3-8\cdot x+6\cdot x^2-x^4)\cdot x^{k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1[(1-x)^3\cdot(x+3)\cdot x^{k-1}]\cdot dx\,.$$

So,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k\cdot(k+1)\cdot(k+2)\cdot(k+4)}=\frac{1}{24}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1[(1-x)^3\cdot(x+3)\cdot x^{k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}[(1-x)^3\cdot(x+3)\cdot x^{k-1}]\cdot dx=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(x+3)\cdot\sum_{k=1}^{n}x^{k-1}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(x+3)\cdot\frac{1-x^n}{1-x}\right]\cdot dx=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1[(1-x)^2\cdot(x+3)\cdot(1-x^n)]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1[(1-x)^2\cdot(x+3)]\cdot dx-\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1[(1-x)^2\cdot(x+3)\cdot x^n]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1(x^3+x^2-5\cdot x+3)\cdot dx=\frac{1}{24}\cdot\left(3\cdot x-5\cdot\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}\right)\Bigg|_0^1$$

$$=\frac{1}{24}\cdot\left(3-5\cdot\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)=\frac{1}{24}\cdot\frac{13}{12}=\frac{13}{288}\,,$$

since:

$$\lim_{n\to\infty}\int_0^1[(1-x)^2\cdot(x+3)\cdot x^n]\cdot dx\,.$$

Example E2.2: The following equality holds:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k+1)\cdot(2\cdot k+2)\cdot(2\cdot k+4)}=\frac{1}{24}\cdot\left(\frac{23}{4}-8\cdot\ln 2\right). \tag{E2.2}$$

Proof**:** Equality (E2.2) is obtained from equality (E2), for:

q=2,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{2\cdot k\cdot(2\cdot k+1)\cdot(2\cdot k+2)\cdot(2\cdot k+4)}=\frac{1}{24}\cdot\left(\frac{3}{2\cdot k}-\frac{8}{2\cdot k+1}+\frac{6}{2\cdot k+2}-\frac{1}{2\cdot k+4}\right)$$

$$=\frac{1}{24}\cdot\int_0^1(3\cdot x^{2\cdot k-1}-8\cdot x^{2\cdot k}+6\cdot x^{2\cdot k+1}-x^{2\cdot k+3})\cdot dx=\frac{1}{24}\cdot\int_0^1[(3-8\cdot x+6\cdot x^2-x^4)\cdot x^{2\cdot k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1[(1-x)^3\cdot(x+3)\cdot x^{2\cdot k-1}]\cdot dx\,.$$

So,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k+1)\cdot(2\cdot k+2)\cdot(2\cdot k+4)}=\frac{1}{24}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1[(1-x)^3\cdot(x+3)\cdot x^{2\cdot k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[\sum_{k=1}^{n}(1-x)^3\cdot(x+3)\cdot x^{k-1}\right]\cdot dx=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(x+3)\cdot\sum_{k=1}^{n}x^{2\cdot k-1}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(x+3)\cdot\frac{1-x^n}{1-x}\right]\cdot dx=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(x+3)\cdot x\cdot(1-x^{2\cdot n})}{1+x}\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\frac{(1-x)^2\cdot x\cdot(x+3)}{1+x}\cdot dx-\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(x+3)\cdot x^{2\cdot n+1}}{1+x}\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\left(\frac{x^4+x^3-5\cdot x^2+3\cdot x}{1+x}\right)\cdot dx=\frac{1}{24}\cdot\left(8\cdot x-5\cdot\frac{x^2}{2}+\frac{x^4}{4}-8\cdot\ln(x+1)\right)\Bigg|_0^1$$

$$=\frac{1}{24}\cdot\left(8-5\cdot\frac{1}{2}+\frac{1}{4}-8\cdot\ln 2\right)=\frac{1}{24}\cdot\left(\frac{23}{4}-8\cdot\ln 2\right),$$

since:

$$\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(x+3)\cdot x^{2\cdot n+1}}{1+x}\cdot dx=0.$$

Example E2.3**:** The following equality holds:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k+1)\cdot(3\cdot k+2)\cdot(3\cdot k+4)}=\frac{1}{48}\cdot\left(\frac{25}{2}-3\cdot\ln 3-5\cdot\sqrt{3}\cdot\frac{\pi}{3}\right). \tag{E2.3}$$

Proof**:** Equality (E2.3) is obtained from equality (E2), for:

q=3,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{3\cdot k\cdot(3\cdot k+1)\cdot(3\cdot k+2)\cdot(3\cdot k+4)}=\frac{1}{24}\cdot\left(\frac{3}{3\cdot k}-\frac{8}{3\cdot k+1}+\frac{6}{3\cdot k+2}-\frac{1}{3\cdot k+4}\right)$$

$$=\frac{1}{24}\cdot\int_0^1(3\cdot x^{3\cdot k-1}-8\cdot x^{3\cdot k}+6\cdot x^{3\cdot k+1}-x^{3\cdot k+3})\cdot dx=\frac{1}{24}\cdot\int_0^1[(3-8\cdot x+6\cdot x^2-x^4)\cdot x^{3\cdot k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1[(1-x)^3\cdot(x+3)\cdot x^{3\cdot k-1}]\cdot dx\,.$$

So,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k+1)\cdot(3\cdot k+2)\cdot(3\cdot k+4)}=\frac{1}{24}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1[(1-x)^3\cdot(x+3)\cdot x^{3\cdot k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}[(1-x)^3\cdot(x+3)\cdot x^{3\cdot k-1}]\cdot dx=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(x+3)\cdot\sum_{k=1}^{n}x^{3\cdot k-1}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(x+3)\cdot x^2\cdot\frac{1-x^{3\cdot n}}{1-x^3}\right]\cdot dx=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(x+3)\cdot x^2\cdot(1-x^{3\cdot n})}{1+x+x^2}\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\frac{(1-x)^2\cdot x^2\cdot(x+3)}{1+x+x^2}\cdot dx-\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(x+3)\cdot x^{3\cdot n+2}}{1+x+x^2}\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\left(\frac{x^5+x^4-5\cdot x^3+3\cdot x^2}{1+x+x^2}\right)\cdot dx=\frac{1}{24}\cdot\int_0^1\left(x^3-6\cdot x+9-\frac{3\cdot x+9}{1+x+x^2}\right)\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\left(x^3-6\cdot x+9-\frac{3}{2}\cdot\frac{2\cdot x+1}{1+x+x^2}+\frac{15}{2}\cdot\frac{1}{x^2+x+1}\right)\cdot dx$$

$$=\frac{1}{24}\cdot\left(9\cdot x-6\cdot\frac{x^2}{2}+\frac{x^4}{4}-\frac{3}{2}\cdot\ln(1+x+x^2)+5\cdot\sqrt{3}\cdot\mathrm{arctg}\left(\frac{2\cdot x+1}{\sqrt{3}}\right)\right)\Bigg|_0^1$$

$$=\frac{1}{24}\cdot\left(9-6\cdot\frac{1}{2}+\frac{1}{4}-\frac{3}{2}\cdot\ln 3+5\cdot\sqrt{3}\cdot\frac{\pi}{6}\right)=\frac{1}{24}\cdot\left(\frac{25}{4}-\frac{3}{2}\cdot\ln 3-5\cdot\sqrt{3}\cdot\frac{\pi}{6}\right)$$

$$=\frac{1}{48}\cdot\left(\frac{25}{2}-3\cdot\ln 3-5\cdot\sqrt{3}\cdot\frac{\pi}{3}\right),$$

since:

$$\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(x+3)\cdot x^{3\cdot n+2}}{1+x+x^2}\cdot dx=0.$$

Example E2.4: The following equality holds:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(4\cdot k)\cdot(4\cdot k+1)\cdot(4\cdot k+2)\cdot(4\cdot k+4)}=\frac{1}{24}\cdot\left(\frac{21}{4}-3\cdot\ln 2-\pi\right).\qquad\text{(E2.4)}$$

Proof: Equality (E2.4) is obtained from equality (E2), for:

q=4,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{4\cdot k\cdot(4\cdot k+1)\cdot(4\cdot k+2)\cdot(4\cdot k+4)}=\frac{1}{24}\cdot\left(\frac{3}{4\cdot k}-\frac{8}{4\cdot k+1}+\frac{6}{4\cdot k+2}-\frac{1}{4\cdot k+4}\right)$$

$$=\frac{1}{24}\cdot\int_0^1(3\cdot x^{4\cdot k-1}-8\cdot x^{4\cdot k}+6\cdot x^{4\cdot k+1}-x^{4\cdot k+3})\cdot dx=\frac{1}{24}\cdot\int_0^1[(3-8\cdot x+6\cdot x^2-x^4)\cdot x^{4\cdot k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1[(1-x)^3\cdot(x+3)\cdot x^{4\cdot k-1}]\cdot dx\ .$$

So,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(4\cdot k)\cdot(4\cdot k+1)\cdot(4\cdot k+2)\cdot(4\cdot k+4)}=\frac{1}{24}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1[(1-x)^3\cdot(x+3)\cdot x^{4\cdot k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}[(1-x)^3\cdot(x+3)\cdot x^{4\cdot k-1}]\cdot dx=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(x+3)\cdot\sum_{k=1}^{n}x^{4\cdot k-1}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(x+3)\cdot x^3\cdot\frac{1-x^{4\cdot n}}{1-x^4}\right]\cdot dx=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(x+3)\cdot x^3\cdot(1-x^{4\cdot n})}{1+x+x^2+x^3}\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\frac{(1-x)^2\cdot x^3\cdot(x+3)}{1+x+x^2+x^3}\cdot dx-\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(x+3)\cdot x^{4\cdot n+3}}{1+x+x^2+x^3}\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\left(\frac{x^6+x^5-5\cdot x^4+3\cdot x^3}{1+x+x^2+x^3}\right)\cdot dx=\frac{1}{24}\cdot\int_0^1\left(x^3-6\cdot x+8-\frac{4}{x+1}+\frac{2\cdot x}{x^2+1}-4\cdot\frac{1}{x^2+1}\right)\cdot dx$$

$$=\frac{1}{24}\cdot\left(8\cdot x-6\cdot\frac{x^2}{2}+\frac{x^4}{4}-4\cdot\ln(x+1)+\ln(x^2+1)-4\cdot\operatorname{arctg}x\right)\Bigg|_0^1$$

$$=\frac{1}{24}\cdot\left(8-6\cdot\frac{1}{2}+\frac{1}{4}-4\cdot\ln 2+2\cdot\ln 2-\frac{\pi}{4}\right)=\frac{1}{24}\cdot\left(\frac{21}{4}-3\cdot\ln 2-\pi\right),$$

since:

$$\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(x+3)\cdot x^{4\cdot n+3}}{1+x+x^2+x^3}\cdot dx=0.$$

Corollary E3: *If* $q\in\mathbf{N}^*$, *then*:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(q\cdot k)\cdot(q\cdot k+1)\cdot(q\cdot k+3)\cdot(q\cdot k+4)}=\frac{1}{12}\cdot\int_0^1\frac{(1-x)^3\cdot(x+1)\cdot x^{q-1}}{1-x^q}\cdot dx. \quad \text{(E3)}$$

Proof**:** Equality (E3) is obtained from Theorem (E), for:

p=1, r=3 and s=4,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{q\cdot k\cdot(q\cdot k+1)\cdot(q\cdot k+3)\cdot(q\cdot k+4)}=\frac{1}{12}\cdot\left(\frac{1}{q\cdot k}-\frac{2}{q\cdot k+1}+\frac{2}{q\cdot k+3}-\frac{1}{q\cdot k+4}\right)$$

$$=\frac{1}{12}\cdot\int_0^1(x^{q\cdot k-1}-2\cdot x^{q\cdot k}+2\cdot x^{q\cdot k+2}-x^{q\cdot k+3})\cdot dx=\frac{1}{12}\cdot\int_0^1[(1-2\cdot x+2\cdot x^3-x^4)\cdot x^{q\cdot k-1}\cdot]dx$$

$$=\frac{1}{12}\cdot\int_0^1[(1-x)^3\cdot(x+1)\cdot x^{q\cdot k-1}]\cdot dx.$$

So,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(q\cdot k)\cdot(q\cdot k+1)\cdot(q\cdot k+3)\cdot(q\cdot k+4)}=\frac{1}{12}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1[(1-x)^3\cdot(x+1)\cdot x^{q\cdot k-1}]\cdot dx$$

$$=\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}[(1-x)^3\cdot(x+1)\cdot x^{q\cdot k-1}]\cdot dx=\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(x+1)\cdot\sum_{k=1}^{n}x^{q\cdot k-1}\right]\cdot dx$$

$$=\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(x+1)\cdot x^{q-1}\cdot\frac{1-x^{n\cdot q}}{1-x^q}\right]\cdot dx$$

$$=\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^3\cdot(x+1)\cdot x^{q-1}}{1-x^q}\cdot dx-\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^3\cdot(x+1)\cdot x^{(n+1)\cdot q-1}}{1-x^q}\cdot dx$$

$$=\frac{1}{12}\cdot\int_0^1\frac{(1-x)^3\cdot(x+1)\cdot x^{q-1}}{1-x^q}\cdot dx,$$

because, according to (Vălcan, 2016, Proposition),

$$\lim_{n\to\infty}\int_0^1\frac{(1-x)^3\cdot(x+1)\cdot x^{(n+1)\cdot q-1}}{1-x^q}\cdot dx=0.$$

Example E3.1**:** The following equality holds:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k\cdot(k+1)\cdot(k+3)\cdot(k+4)}=\frac{5}{144}. \quad \text{(E3.1)}$$

Proof**:** Equality (E3.1) is obtained from equality (E3), for:

q=1,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{k\cdot(k+1)\cdot(k+3)\cdot(k+4)}=\frac{1}{12}\cdot\left(\frac{1}{k}-\frac{2}{k+1}+\frac{2}{k+3}-\frac{1}{k+4}\right)$$

$$=\frac{1}{12}\cdot\int_0^1(x^{k-1}-2\cdot x^k+2\cdot x^{k+2}-x^{k+3})\cdot dx=\frac{1}{12}\cdot\int_0^1[(1-2\cdot x+2\cdot x^3-x^4)\cdot x^{k-1}]\cdot dx$$

$$=\frac{1}{12}\cdot\int_0^1[(1-x)^3\cdot(x+1)\cdot x^{k-1}]\cdot dx\,.$$

So,

$$\lim_{n\to\infty}\sum_{k=1}^n\frac{1}{k\cdot(k+1)\cdot(k+3)\cdot(k+4)}=\frac{1}{12}\cdot\lim_{n\to\infty}\sum_{k=1}^n\int_0^1[(1-x)^3\cdot(x+1)\cdot x^{k-1}]\cdot dx$$

$$=\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^n[(1-x)^3\cdot(x+1)\cdot x^{k-1}]\cdot dx=\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(x+1)\cdot\sum_{k=1}^n x^{k-1}\right]\cdot dx$$

$$=\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(x+1)\cdot\frac{1-x^n}{1-x}\right]\cdot dx=\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1[(1-x)^2\cdot(x+1)\cdot(1-x^n)]\cdot dx$$

$$=\frac{1}{12}\cdot\int_0^1[(1-x)^2\cdot(x+1)]\cdot dx-\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1[(1-x)^2\cdot(x+1)\cdot x^n]\cdot dx$$

$$=\frac{1}{12}\cdot\int_0^1(x^3-x^2-x+1)\cdot dx=\frac{1}{12}\cdot\left(x-\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{4}\right)\Bigg|_0^1$$

$$=\frac{1}{12}\cdot\left(1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}\right)=\frac{1}{12}\cdot\frac{5}{12}=\frac{5}{144}\,,$$

since:

$$\lim_{n\to\infty}\int_0^1[(1-x)^2\cdot(x+1)\cdot x^n]\cdot dx=0.$$

Example E3.2: The following equality holds:

$$\lim_{n\to\infty}\sum_{k=1}^n\frac{1}{(2\cdot k)\cdot(2\cdot k+1)\cdot(2\cdot k+3)\cdot(2\cdot k+4)}=\frac{1}{144}\,. \tag{E3.2}$$

Proof: Equality (E3.2) is obtained from equality (E3), for:

q=2,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{2\cdot k\cdot(2\cdot k+1)\cdot(2\cdot k+3)\cdot(2\cdot k+4)}=\frac{1}{12}\cdot\left(\frac{1}{2\cdot k}-\frac{2}{2\cdot k+1}+\frac{2}{2\cdot k+3}-\frac{1}{2\cdot k+4}\right)$$

$$=\frac{1}{12}\cdot\int_0^1(x^{2\cdot k-1}-2\cdot x^{2\cdot k}+2\cdot x^{2\cdot k+2}-x^{2\cdot k+3})\cdot dx=\frac{1}{12}\cdot\int_0^1[(1-2\cdot x+2\cdot x^3-x^4)\cdot x^{k-1}]\cdot dx$$

$$=\frac{1}{12}\cdot\int_0^1[(1-x)^3\cdot(x+1)\cdot x^{k-1}]\cdot dx\,.$$

So,

$$\lim_{n\to\infty}\sum_{k=1}^n\frac{1}{(2\cdot k)\cdot(2\cdot k+1)\cdot(2\cdot k+3)\cdot(2\cdot k+4)}=\frac{1}{12}\cdot\lim_{n\to\infty}\sum_{k=1}^n\int_0^1[(1-x)^3\cdot(x+1)\cdot x^{2k-1}]\cdot dx$$

$$=\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^n[(1-x)^3\cdot(x+1)\cdot x^{k-1}]\cdot dx=\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(x+1)\cdot\sum_{k=1}^n x^{2\cdot k-1}\right]\cdot dx$$

$$=\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(x+1)\cdot x\cdot\frac{1-x^{2\cdot n}}{1-x^2}\right]\cdot dx=\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(x+1)\cdot x\cdot(1-x^{2\cdot n})}{1+x}\cdot dx$$

$$=\frac{1}{12}\cdot\int_0^1[(1-x)^2\cdot x]\cdot dx-\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1[(1-x)^2\cdot x^{2\cdot n+1}]\cdot dx=\frac{1}{12}\cdot\int_0^1(x^3-2x^2+x)\cdot dx$$

$$=\frac{1}{12}\cdot\left(\frac{x^2}{2}-2\cdot\frac{x^3}{3}+\frac{x^4}{4}\right)\Bigg|_0^1=\frac{1}{12}\cdot\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right)=\frac{1}{12}\cdot\frac{1}{12}=\frac{1}{144},$$

since:

$$\lim_{n\to\infty}\int_0^1[(1-x)^2\cdot x^{2\cdot n+1}]\cdot dx=0.$$

***Example E3.3*:** The following equality holds:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k+1)\cdot(3\cdot k+3)\cdot(3\cdot k+4)}=\frac{1}{24}\cdot\left(\frac{31}{6}-3\cdot\ln 3-\frac{\pi\cdot\sqrt{3}}{3}\right).\tag{E3.3}$$

***Proof*:** Equality (E3.3) is obtained from equality (E3), for:

q=3,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{3\cdot k\cdot(3\cdot k+1)\cdot(3\cdot k+3)\cdot(3\cdot k+4)}=\frac{1}{12}\cdot\left(\frac{1}{3\cdot k}-\frac{2}{3\cdot k+1}+\frac{2}{3\cdot k+3}-\frac{1}{3\cdot k+4}\right)$$

$$=\frac{1}{12}\cdot\int_0^1(x^{3\cdot k-1}-2\cdot x^{3\cdot k}+2\cdot x^{3\cdot k+2}-x^{3\cdot k+3})\cdot dx=\frac{1}{12}\cdot\int_0^1[(1-2\cdot x+2\cdot x^3-x^4)\cdot x^{3\cdot k-1}]\cdot dx$$

$$=\frac{1}{12}\cdot\int_0^1[(1-x)^3\cdot(x+1)\cdot x^{3\cdot k-1}]\cdot dx\,.$$

So,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k+1)\cdot(3\cdot k+3)\cdot(3\cdot k+4)}=\frac{1}{12}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1[(1-x)^3\cdot(x+1)\cdot x^{3k-1}]\cdot dx$$

$$=\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}[(1-x)^3\cdot(x+1)\cdot x^{3\cdot k-1}]\cdot dx=\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(x+1)\cdot\sum_{k=1}^{n}x^{3\cdot k-1}\right]\cdot dx$$

$$=\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(x+1)\cdot x^2\cdot\frac{1-x^{3\cdot n}}{1-x^3}\right]\cdot dx=\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(x+1)\cdot x^2\cdot(1-x^{3\cdot n})}{1+x+x^2}\cdot dx$$

$$=\frac{1}{12}\cdot\int_0^1\frac{(1-x)^2\cdot(x+1)\cdot x^2}{1+x+x^2}\cdot dx-\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(x+1)\cdot x^{3\cdot n+2}}{1+x+x^2}\cdot dx$$

$$=\frac{1}{12}\cdot\int_0^1\left(x^3-2x^2+3-\frac{3}{2}\cdot\frac{2\cdot x+1}{x^2+x+1}-\frac{3}{2}\cdot\frac{1}{x^2+x+1}\right)\cdot dx$$

$$=\frac{1}{12}\cdot\left(3\cdot x-2\cdot\frac{x^3}{3}+\frac{x^4}{4}-\frac{3}{2}\cdot\ln(x^2+x+1)-\sqrt{3}\cdot\mathrm{arctg}\left(\frac{2\cdot x+1}{\sqrt{3}}\right)\right)\Bigg|_0^1$$

$$=\frac{1}{12}\cdot\left(3-\frac{2}{3}+\frac{1}{4}-\frac{3}{2}\cdot\ln 3-\frac{\pi\cdot\sqrt{3}}{6}\right)=\frac{1}{24}\cdot\left(\frac{31}{6}-3\cdot\ln 3-\frac{\pi\cdot\sqrt{3}}{3}\right),$$

since:

$$\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(x+1)\cdot x^{3\cdot n+2}}{1+x+x^2}\cdot dx=0.$$

***Example E3.4*:** The following equality holds:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(4\cdot k)\cdot(4\cdot k+1)\cdot(4\cdot k+3)\cdot(4\cdot k+4)}=\frac{1}{24}\cdot\left(\frac{19}{6}-\pi\right). \tag{E3.4}$$

Proof: Equality (E3.4) is obtained from equality (E3), for:

q=4,

or by direct calculation. Thus, for every $k\in\mathbf{N}^{*}$,

$$\frac{1}{4\cdot k\cdot(4\cdot k+1)\cdot(4\cdot k+3)\cdot(4\cdot k+4)}=\frac{1}{12}\cdot\left(\frac{1}{4\cdot k}-\frac{2}{4\cdot k+1}+\frac{2}{4\cdot k+3}-\frac{1}{4\cdot k+4}\right)$$

$$=\frac{1}{12}\cdot\int_0^1(x^{4\cdot k-1}-2\cdot x^{4\cdot k}+2\cdot x^{4\cdot k+2}-x^{4\cdot k+3})\cdot dx=\frac{1}{12}\cdot\int_0^1[(1-2\cdot x+2\cdot x^3-x^4)\cdot x^{4\cdot k-1}]\cdot dx$$

$$=\frac{1}{12}\cdot\int_0^1[(1-x)^3\cdot(x+1)\cdot x^{4\cdot k-1}]\cdot dx\,.$$

So,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(4\cdot k)\cdot(4\cdot k+1)\cdot(4\cdot k+3)\cdot(4\cdot k+4)}=\frac{1}{12}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1[(1-x)^3\cdot(x+1)\cdot x^{4k-1}]\cdot dx$$

$$=\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}[(1-x)^3\cdot(x+1)\cdot x^{4\cdot k-1}]\cdot dx=\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(x+1)\cdot\sum_{k=1}^{n}x^{4\cdot k-1}\right]\cdot dx$$

$$=\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(x+1)\cdot x^3\cdot\frac{1-x^{4\cdot n}}{1-x^4}\right]\cdot dx=\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(x+1)\cdot x^3\cdot(1-x^{4\cdot n})}{1+x+x^2+x^3}\cdot dx$$

$$=\frac{1}{12}\cdot\int_0^1\frac{(1-x)^2\cdot(x+1)\cdot x^3}{1+x+x^2+x^3}\cdot dx-\frac{1}{12}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(x+1)\cdot x^{4\cdot n+3}}{1+x+x^2+x^3}\cdot dx$$

$$=\frac{1}{12}\cdot\int_0^1\left(x^3-2x^2+2-2\cdot\frac{1}{x^2+1}\right)\cdot dx=\frac{1}{12}\cdot\left(2\cdot x-2\cdot\frac{x^3}{3}+\frac{x^4}{4}-2\cdot\operatorname{arctg}x\right)\Big|_0^1$$

$$=\frac{1}{12}\cdot\left(2-\frac{2}{3}+\frac{1}{4}-2\cdot\frac{\pi}{4}\right)=\frac{1}{24}\cdot\left(\frac{19}{6}-\pi\right),$$

since:

$$\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(x+1)\cdot x^{4\cdot n+3}}{1+x+x^2+x^3}\cdot dx=0.$$

Corollary E4: *If $q\in\mathbf{N}^{*}$, then*:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(q\cdot k)\cdot(q\cdot k+2)\cdot(q\cdot k+3)\cdot(q\cdot k+4)}=\frac{1}{24}\cdot\int_0^1\frac{(1-x)^3\cdot(3\cdot x+1)\cdot x^{q-1}}{1-x^q}\cdot dx\,. \tag{E4}$$

Proof: Equality (E4) is obtained from Theorem (E), for:

p=2, r=3 and s=4,

or by direct calculation. Thus, for every $k\in\mathbf{N}^{*}$,

$$\frac{1}{q\cdot k\cdot(q\cdot k+2)\cdot(q\cdot k+3)\cdot(q\cdot k+4)}=\frac{1}{24}\cdot\left(\frac{1}{q\cdot k}-\frac{6}{q\cdot k+1}+\frac{8}{q\cdot k+3}-\frac{3}{q\cdot k+4}\right)$$

$$=\frac{1}{24}\cdot\int_0^1(x^{q\cdot k-1}-6\cdot x^{q\cdot k}+8\cdot x^{q\cdot k+2}-3\cdot x^{q\cdot k+3})\cdot dx=\frac{1}{24}\cdot\int_0^1[(1-6\cdot x+8\cdot x^3-3\cdot x^4)\cdot x^{q\cdot k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1[(1-x)^3\cdot(3\cdot x+1)\cdot x^{q\cdot k-1}]\cdot dx\,.$$

So,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(q\cdot k)\cdot(q\cdot k+2)\cdot(q\cdot k+3)\cdot(q\cdot k+4)}=\frac{1}{24}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1[(1-x)^3\cdot(3\cdot x+1)\cdot x^{q\cdot k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}[(1-x)^3\cdot(3\cdot x+1)\cdot x^{q\cdot k-1}]\cdot dx=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(3\cdot x+1)\cdot\sum_{k=1}^{n}x^{q\cdot k-1}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(3\cdot x+1)\cdot x^{q-1}\cdot\frac{1-x^{n\cdot q}}{1-x^q}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^3\cdot(3\cdot x+1)\cdot x^{q-1}}{1-x^q}\cdot dx-\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^3\cdot(3\cdot x+1)\cdot x^{(n+1)\cdot q-1}}{1-x^q}\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\frac{(1-x)^3\cdot(3\cdot x+1)\cdot x^{q-1}}{1-x^q}\cdot dx,$$

because, according to (Vălcan, 2016, Proposition),

$$\lim_{n\to\infty}\int_0^1\frac{(1-x)^3\cdot(3\cdot x+1)\cdot x^{(n+1)\cdot q-1}}{1-x^q}\cdot dx=0.$$

***Example E4.1*:** The following equality holds:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k\cdot(k+2)\cdot(k+3)\cdot(k+4)}=\frac{7}{288}.\qquad\text{(E4.1)}$$

***Proof*:** Equality (E4.1) is obtained from equality (E4), for:

q=1,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{k\cdot(k+2)\cdot(k+3)\cdot(k+4)}=\frac{1}{24}\cdot\left(\frac{1}{k}-\frac{6}{k+2}+\frac{8}{k+3}-\frac{3}{k+4}\right)$$

$$=\frac{1}{24}\cdot\int_0^1(x^{k-1}-6\cdot x^{k+1}+8\cdot x^{k+2}-3\cdot x^{k+3})\cdot dx=\frac{1}{24}\cdot\int_0^1[(1-6\cdot x^2+8\cdot x^3-3\cdot x^4)\cdot x^{k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1[(1-x)^3\cdot(3\cdot x+1)\cdot x^{k-1}]\cdot dx.$$

So,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k\cdot(k+2)\cdot(k+3)\cdot(k+4)}=\frac{1}{24}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1[(1-x)^3\cdot(3\cdot x+1)\cdot x^{k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}[(1-x)^3\cdot(3\cdot x+1)\cdot x^{k-1}]\cdot dx=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(3\cdot x+1)\cdot\sum_{k=1}^{n}x^{k-1}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(3\cdot x+1)\cdot\frac{1-x^n}{1-x}\right]\cdot dx=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1[(1-x)^2\cdot(3\cdot x+1)\cdot(1-x^n)]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1[(1-x)^2\cdot(3\cdot x+1)]\cdot dx-\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1[(1-x)^2\cdot(3\cdot x+1)\cdot x^n]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1(3\cdot x^3-5\cdot x^2+x+1)\cdot dx=\frac{1}{24}\cdot\left(x+\frac{x^2}{2}-5\cdot\frac{x^3}{3}+3\cdot\frac{x^4}{4}\right)\Bigg|_0^1$$

$$=\frac{1}{24}\cdot\left(1+\frac{1}{2}-\frac{5}{3}+\frac{3}{4}\right)=\frac{1}{24}\cdot\frac{7}{12}=\frac{7}{288},$$

since:

$$\lim_{n\to\infty}\int_0^1 [(1-x)^2\cdot(3\cdot x+1)\cdot x^n]\cdot dx = 0.$$

Example E4.2: The following equality holds:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k+2)\cdot(2\cdot k+3)\cdot(2\cdot k+4)}=\frac{1}{24}\cdot\left(8\cdot\ln 2-\frac{65}{12}\right). \qquad \text{(E4.2)}$$

Proof: Equality (E4.2) is obtained from equality (E4), for:

q=2,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{2\cdot k\cdot(2\cdot k+2)\cdot(2\cdot k+3)\cdot(2\cdot k+4)}=\frac{1}{24}\cdot\left(\frac{1}{2\cdot k}-\frac{6}{2\cdot k+2}+\frac{8}{2\cdot k+3}-\frac{3}{2\cdot k+4}\right)$$

$$=\frac{1}{24}\cdot\int_0^1(x^{2\cdot k-1}-6\cdot x^{2\cdot k+1}+8\cdot x^{2\cdot k+2}-3\cdot x^{2\cdot k+3})\cdot dx=\frac{1}{24}\cdot\int_0^1[(1-6\cdot x^2+8\cdot x^3-3\cdot x^4)\cdot x^{2\cdot k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1[(1-x)^3\cdot(3\cdot x+1)\cdot x^{2\cdot k-1}]\cdot dx .$$

So,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k+2)\cdot(2\cdot k+3)\cdot(2\cdot k+4)}=\frac{1}{24}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1[(1-x)^3\cdot(3\cdot x+1)\cdot x^{2\cdot k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}[(1-x)^3\cdot(3\cdot x+1)\cdot x^{2\cdot k-1}]\cdot dx=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(3\cdot x+1)\cdot\sum_{k=1}^{n}x^{2\cdot k-1}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(3\cdot x+1)\cdot x\cdot\frac{1-x^{2\cdot n}}{1-x^2}\right]\cdot dx=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(3\cdot x+1)\cdot x\cdot(1-x^n)}{1+x}\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\frac{(1-x)^2\cdot x\cdot(3\cdot x+1)}{1+x}\cdot dx - \frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(3\cdot x+1)\cdot x^{n+1}}{1+x}\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\left(\frac{3\cdot x^4-5\cdot x^3+x^2+x}{x+1}\right)\cdot dx=\frac{1}{24}\cdot\left(-8\cdot x+9\cdot\frac{x^2}{2}-8\cdot\frac{x^3}{3}+3\cdot\frac{x^4}{4}+8\cdot\ln(x+1)\right)\Bigg|_0^1$$

$$=\frac{1}{24}\cdot\left(-8+\frac{9}{2}-\frac{8}{3}+\frac{3}{4}+8\cdot\ln 2\right)=\frac{1}{24}\cdot\left(8\cdot\ln 2-\frac{65}{12}\right),$$

since:

$$\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(3\cdot x+1)\cdot x^{n+1}}{1+x}\cdot dx=0$$

Example E4.3: The following equality holds:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k+2)\cdot(3\cdot k+3)\cdot(3\cdot k+4)}=\frac{1}{48}\cdot\left(\frac{49}{6}-9\cdot\ln 3+\frac{\pi\cdot\sqrt{3}}{3}\right). \qquad \text{(E4.3)}$$

Proof: Equality (E4.3) is obtained from equality (E4), for:

q=3,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{3\cdot k\cdot(3\cdot k+2)\cdot(3\cdot k+3)\cdot(3\cdot k+4)}=\frac{1}{24}\cdot\left(\frac{1}{3\cdot k}-\frac{6}{3\cdot k+2}+\frac{8}{3\cdot k+3}-\frac{3}{3\cdot k+4}\right)$$

$$=\frac{1}{24}\cdot\int_0^1(x^{3\cdot k-1}-6\cdot x^{3\cdot k+1}+8\cdot x^{3\cdot k+2}-3\cdot x^{3\cdot k+3})\cdot dx=\frac{1}{24}\cdot\int_0^1[(1-6\cdot x^2+8\cdot x^3-3\cdot x^4)\cdot x^{3\cdot k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1[(1-x)^3\cdot(3\cdot x+1)\cdot x^{3\cdot k-1}]\cdot dx\,.$$

So,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k+2)\cdot(3\cdot k+3)\cdot(3\cdot k+4)}=\frac{1}{24}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1[(1-x)^3\cdot(3\cdot x+1)\cdot x^{3\cdot k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}[(1-x)^3\cdot(3\cdot x+1)\cdot x^{3\cdot k-1}]\cdot dx=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(3\cdot x+1)\cdot\sum_{k=1}^{n}x^{3\cdot k-1}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(3\cdot x+1)\cdot x^2\cdot\frac{1-x^{3\cdot n}}{1-x^3}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(3\cdot x+1)\cdot x^2\cdot(1-x^{3\cdot n})}{1+x+x^2}\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\frac{(1-x)^2\cdot x^2\cdot(3\cdot x+1)}{1+x+x^2}\cdot dx-\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(3\cdot x+1)\cdot x^{3\cdot n+2}}{1+x+x^2}\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\left(\frac{3\cdot x^5-5\cdot x^4+x^3+x^2}{x^2+x+1}\right)\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\left(3\cdot x^3-8\cdot x^2+6\cdot x+3-\frac{9}{2}\cdot\frac{2\cdot x+1}{x^2+x+1}+\frac{3}{2}\cdot\frac{1}{x^2+x+1}\right)\cdot dx$$

$$=\frac{1}{24}\cdot\left(3\cdot x+6\cdot\frac{x^2}{2}-8\cdot\frac{x^3}{3}+3\cdot\frac{x^4}{4}-\frac{9}{2}\cdot\ln(x^2+x+1)+\sqrt{3}\cdot\operatorname{arctg}\left(\frac{2\cdot x+1}{\sqrt{3}}\right)\right)\Bigg|_0^1$$

$$=\frac{1}{24}\cdot\left(3+\frac{6}{2}-\frac{8}{3}+\frac{3}{4}-\frac{9}{2}\cdot\ln 3+\frac{\pi\cdot\sqrt{3}}{6}\right)=\frac{1}{48}\cdot\left(\frac{49}{6}-9\cdot\ln 3+\frac{\pi\cdot\sqrt{3}}{3}\right),$$

since:

$$\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(3\cdot x+1)\cdot x^{3\cdot n+2}}{1+x+x^2}\cdot dx=0.$$

***Example E4.4*:** The following equality holds:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(4\cdot k)\cdot(4\cdot k+2)\cdot(4\cdot k+3)\cdot(4\cdot k+4)}=\frac{1}{24}\cdot\left(\frac{13}{12}+3\cdot\ln 2-\pi\right). \tag{E4.4}$$

***Proof*:** Equality (E4.4) is obtained from equality (E4), for:

$q=4$,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{4\cdot k\cdot(4\cdot k+2)\cdot(4\cdot k+3)\cdot(4\cdot k+4)}=\frac{1}{24}\cdot\left(\frac{1}{4\cdot k}-\frac{6}{4\cdot k+2}+\frac{8}{4\cdot k+3}-\frac{3}{4\cdot k+4}\right)$$

$$=\frac{1}{24}\cdot\int_0^1(x^{4\cdot k-1}-6\cdot x^{4\cdot k+1}+8\cdot x^{4\cdot k+2}-3\cdot x^{4\cdot k+3})\cdot dx=\frac{1}{24}\cdot\int_0^1[(1-6\cdot x^2+8\cdot x^3-3\cdot x^4)\cdot x^{4\cdot k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1[(1-x)^3\cdot(3\cdot x+1)\cdot x^{4\cdot k-1}]\cdot dx\,.$$

So,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(4\cdot k)\cdot(4\cdot k+2)\cdot(4\cdot k+3)\cdot(4\cdot k+4)}=\frac{1}{24}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1[(1-x)^3\cdot(3\cdot x+1)\cdot x^{4\cdot k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}[(1-x)^3\cdot(3\cdot x+1)\cdot x^{4\cdot k-1}]\cdot dx=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(3\cdot x+1)\cdot\sum_{k=1}^{n}x^{4\cdot k-1}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot(3\cdot x+1)\cdot x^3\cdot\frac{1-x^{4\cdot n}}{1-x^4}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(3\cdot x+1)\cdot x^3\cdot(1-x^{4\cdot n})}{1+x+x^2+x^3}\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\frac{(1-x)^2\cdot x^3\cdot(3\cdot x+1)}{1+x+x^2+x^3}\cdot dx-\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(3\cdot x+1)\cdot x^{4\cdot n+3}}{1+x+x^2+x^3}\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\left(\frac{3\cdot x^6-5\cdot x^5+x^4+x^3}{x^3+x^2+x+1}\right)\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\left(3\cdot x^3-8\cdot x^2+6\cdot x+4\cdot\frac{1}{x+1}-4\cdot\frac{1}{x^2+1}-2\cdot\frac{x}{x^2+1}\right)\cdot dx$$

$$=\frac{1}{24}\cdot\left(6\cdot\frac{x^2}{2}-8\cdot\frac{x^3}{3}+3\cdot\frac{x^4}{4}+4\cdot\ln(x+1)-4\cdot\text{arctg}x-2\cdot\frac{1}{2}\cdot\ln(x^2+1)\right)\Bigg|_0^1$$

$$=\frac{1}{24}\cdot\left(\frac{6}{2}-\frac{8}{3}+\frac{3}{4}+4\cdot\ln 2-\pi-\ln 2\right)=\frac{1}{24}\cdot\left(\frac{13}{12}+3\cdot\ln 2-\pi\right),$$

since:

$$\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(3\cdot x+1)\cdot x^{4\cdot n+3}}{1+x+x^2+x^3}\cdot dx=0.$$

We now move on to another category of sequences.

F) ***Theorem F*****:** *If p, q, r, s, t$\in$**N*** and p<r<s<t, then*:

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(q\cdot k+p)\cdot(q\cdot k+r)\cdot(q\cdot k+s)\cdot(q\cdot k+t)}=\alpha\cdot\lim_{n\to\infty}\int_0^1\left[\frac{F(x)}{1-x^q}\right]\cdot dx, \qquad \text{(F)}$$

where:

$F(x)=(\alpha_1-\alpha_2\cdot x^{r-p}+\alpha_3\cdot x^{s-p}-\alpha_4\cdot x^{t-p})\cdot x^{p-1}$,

$$\alpha=\frac{1}{(r-p)\cdot(s-p)\cdot(t-p)\cdot(s-r)\cdot(t-r)\cdot(t-s)},$$

$\alpha_1=(s-r)\cdot(t-r)\cdot(t-s)$, $\alpha_2=(s-p)\cdot(t-p)\cdot(t-s)$, $\alpha_3=(r-p)\cdot(t-p)\cdot(t-r)$,
$\alpha_4=(r-p)\cdot(s-p)\cdot(s-r)$.

Proof**:** For every k$\in$**N***,

$$\frac{1}{(q\cdot k+p)\cdot(q\cdot k+r)\cdot(q\cdot k+s)\cdot(q\cdot k+t)}=\alpha\cdot\left(\frac{\alpha_1}{q\cdot k+p}-\frac{\alpha_2}{q\cdot k+r}+\frac{\alpha_3}{q\cdot k+s}-\frac{\alpha_4}{q\cdot k+s}\right)$$

$$=\alpha\cdot\int_0^1(\alpha_1\cdot x^{q\cdot k+p-1}-\alpha_2\cdot x^{q\cdot k+r-1}+\alpha_3\cdot x^{q\cdot k+s-1}-\alpha_4\cdot x^{q\cdot k+t-1})\cdot dx$$

$$=\alpha\cdot\int_0^1(\alpha_1-\alpha_2\cdot x^{r-p}+\alpha_3\cdot x^{s-p}-\alpha_4\cdot x^{t-p})\cdot x^{q\cdot k+p-1}\cdot dx,$$

where:

$$\alpha=\frac{1}{(r-p)\cdot(s-p)\cdot(t-p)\cdot(s-r)\cdot(t-r)\cdot(t-s)},$$

$\alpha_1=(s-r)\cdot(t-r)\cdot(t-s)$, $\alpha_2=(s-p)\cdot(t-p)\cdot(t-s)$, $\alpha_3=(r-p)\cdot(t-p)\cdot(t-r)$, $\alpha_4=(r-p)\cdot(s-p)\cdot(s-r)$.

So,

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(q\cdot k+p)\cdot(q\cdot k+r)\cdot(q\cdot k+s)\cdot(q\cdot k+t)}=$$

$$=\alpha\cdot\lim_{n\to\infty}\sum_{k=0}^{n}\int_0^1[(\alpha_1-\alpha_2\cdot x^{r-p}+\alpha_3\cdot x^{s-p}-\alpha_4\cdot x^{t-p})\cdot x^{q\cdot k+p-1}]\cdot dx$$

$$=\alpha\cdot\lim_{n\to\infty}\int_0^1\sum_{k=0}^{n}[(\alpha_1-\alpha_2\cdot x^{r-p}+\alpha_3\cdot x^{s-p}-\alpha_4\cdot x^{t-p})\cdot x^{q\cdot k+p-1}]\cdot dx$$

$$=\alpha\cdot\lim_{n\to\infty}\int_0^1\left[(\alpha_1-\alpha_2\cdot x^{r-p}+\alpha_3\cdot x^{s-p}-\alpha_4\cdot x^{t-p})\cdot x^{p-1}\cdot\sum_{k=0}^{n}x^{q\cdot k}\right]\cdot dx$$

$$=\alpha\cdot\lim_{n\to\infty}\int_0^1\left[(\alpha_1-\alpha_2\cdot x^{r-p}+\alpha_3\cdot x^{s-p}-\alpha_4\cdot x^{t-p})\cdot x^{p-1}\cdot\frac{1-x^{(n+1)\cdot q}}{1-x^q}\right]\cdot dx$$

$$=\alpha\cdot\lim_{n\to\infty}\int_0^1\left[\frac{(\alpha_1-\alpha_2\cdot x^{r-p}+\alpha_3\cdot x^{s-p}-\alpha_4\cdot x^{t-p})\cdot x^{p-1}}{1-x^q}\right]\cdot dx$$

$$-\alpha\cdot\lim_{n\to\infty}\int_0^1\left[\frac{(\alpha_1-\alpha_2\cdot x^{r-p}+\alpha_3\cdot x^{s-p}-\alpha_4\cdot x^{t-p})\cdot x^{(n+1)\cdot q+p-1}}{1-x^q}\right]\cdot dx$$

$$=\alpha\cdot\int_0^1\left[\frac{(\alpha_1-\alpha_2\cdot x^{r-p}+\alpha_3\cdot x^{s-p}-\alpha_4\cdot x^{t-p})\cdot x^{p-1}}{1-x^q}\right]\cdot dx\,,$$

because, according to (Vălcan, 2016, Proposition),

$$\lim_{n\to\infty}\int_0^1\left[\frac{(\alpha_1-\alpha_2\cdot x^{r-p}+\alpha_3\cdot x^{s-p}-\alpha_4\cdot x^{t-p})\cdot x^{(n+1)\cdot q+p-1}}{1-x^q}\right]\cdot dx=0.$$

Here, too, the following remark is necessary:

Remark F0: *A simple calculation shows us that*:

α_1-α_2+α_3-α_4=0,

which tells us that none of the above integrals is not improper.

Corollary F1: *If* $q\in\mathbf{N}^*$, *then*:

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(q\cdot k+1)\cdot(q\cdot k+2)\cdot(q\cdot k+3)\cdot(q\cdot k+4)}=\frac{1}{6}\cdot\int_0^1\frac{(1-x)^3}{1-x^q}\cdot dx\,. \quad (F1)$$

Proof: Equality (F1) is obtained from Theorem (F), for:

p=1, r=2, s=3 and t=4

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{(q\cdot k+1)\cdot(q\cdot k+2)\cdot(q\cdot k+3)\cdot(q\cdot k+4)}=\frac{1}{6}\cdot\left(\frac{1}{q\cdot k+1}-\frac{3}{q\cdot k+2}+\frac{3}{q\cdot k+3}-\frac{1}{q\cdot k+4}\right)$$

$$=\frac{1}{6}\cdot\int_0^1(x^{q\cdot k}-3\cdot x^{q\cdot k+1}+3\cdot x^{q\cdot k+2}-x^{q\cdot k+3})\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1[(1-3\cdot x+3\cdot x^2-x^3)\cdot x^{q\cdot k}]\cdot dx\,.$$

So,

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(q\cdot k+1)\cdot(q\cdot k+2)\cdot(q\cdot k+3)\cdot(q\cdot k+4)}=$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\sum_{k=0}^{n}\int_0^1[(1-3\cdot x+3\cdot x^2-x^3)\cdot x^{q\cdot k}]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=0}^{n}[(1-3\cdot x+3\cdot x^2-x^3)\cdot x^{q\cdot k}]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-3\cdot x+3\cdot x^2-x^3)\cdot\sum_{k=0}^{n}x^{q\cdot k}\right]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-3\cdot x+3\cdot x^2-x^3)\cdot\frac{1-x^{(n+1)\cdot q}}{1-x^q}\right]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\frac{1-3\cdot x+3\cdot x^2-x^3}{1-x^q}\cdot dx-\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[\frac{(1-3\cdot x+3\cdot x^2-x^3)\cdot x^{(n+1)\cdot q}}{1-x^q}\right]\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1\frac{(1-x)^3}{1-x^q}\cdot dx,$$

because, according to (Vălcan, 2016, Proposition),

$$\lim_{n\to\infty}\int_0^1\left[\frac{(1-3\cdot x+3\cdot x^2-x^3)\cdot x^{(n+1)\cdot q}}{1-x^q}\right]\cdot dx=0.$$

Example F1.1: The following equality holds:

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(k+1)\cdot(k+2)\cdot(k+3)\cdot(k+4)}=\frac{1}{18}. \qquad \text{(F1.1)}$$

Proof: Equality (F1.1) is obtained from equality (F1), for:

q=1,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{(k+1)\cdot(k+2)\cdot(k+3)\cdot(k+4)}=\frac{1}{6}\cdot\left(\frac{1}{k+1}-\frac{3}{k+2}+\frac{3}{k+3}-\frac{1}{k+4}\right)$$

$$=\frac{1}{6}\cdot\int_0^1(x^k-3\cdot x^{k+1}+3\cdot x^{k+2}-x^{k+3})\cdot dx=\frac{1}{6}\cdot\int_0^1[(1-3\cdot x+3\cdot x^2-x^3)\cdot x^k]\cdot dx.$$

So,

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(k+1)\cdot(k+2)\cdot(k+3)\cdot(k+4)}=\frac{1}{6}\cdot\lim_{n\to\infty}\sum_{k=0}^{n}\int_0^1[(1-3\cdot x+3\cdot x^2-x^3)\cdot x^k]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=0}^{n}[(1-3\cdot x+3\cdot x^2-x^3)\cdot x^k]\cdot dx=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-3\cdot x+3\cdot x^2-x^3)\cdot\sum_{k=0}^{n}x^k\right]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-3\cdot x+3\cdot x^2-x^3)\cdot\frac{1-x^{n+1}}{1-x}\right]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\frac{1-3\cdot x+3\cdot x^2-x^3}{1-x}\cdot dx-\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[\frac{(1-3\cdot x+3\cdot x^2-x^3)\cdot x^{n+1}}{1-x}\right]\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1\frac{(1-x)^3}{1-x}\cdot dx=\frac{1}{6}\cdot\int_0^1(1-2\cdot x+x^2)\cdot dx=\frac{1}{6}\cdot\left(x-2\cdot\frac{x^2}{2}+\frac{x^3}{3}\right)\Bigg|_0^1$$

$$=\frac{1}{6}\cdot\left(1-1+\frac{1}{3}\right)\Bigg|_0^1=\frac{1}{18},$$

since,

$$\lim_{n\to\infty}\int_0^1\left[\frac{(1-3\cdot x+3\cdot x^2-x^3)\cdot x^{n+1}}{1-x}\right]\cdot dx=0.$$

Example F1.2: The following equality holds:

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(2\cdot k+1)\cdot(2\cdot k+2)\cdot(2\cdot k+3)\cdot(2\cdot k+4)}=\frac{1}{6}\cdot\left(4\cdot\ln 2-\frac{5}{2}\right). \tag{F1.2}$$

***Proof*: Equality (F1.2) is obtained from equality (F1), for:**

q=2,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{(2\cdot k+1)\cdot(2\cdot k+2)\cdot(2\cdot k+3)\cdot(2\cdot k+4)}=\frac{1}{6}\cdot\left(\frac{1}{2\cdot k+1}-\frac{3}{2\cdot k+2}+\frac{3}{2\cdot k+3}-\frac{1}{2\cdot k+4}\right)$$

$$=\frac{1}{6}\cdot\int_0^1(x^{2\cdot k}-3\cdot x^{2\cdot k+1}+3\cdot x^{2\cdot k+2}-x^{2\cdot k+3})\cdot dx=\frac{1}{6}\cdot\int_0^1[(1-3\cdot x+3\cdot x^2-x^3)\cdot x^{2\cdot k}]\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1[(1-x)^3\cdot x^{2\cdot k}]\cdot dx.$$

So,

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(2\cdot k+1)\cdot(2\cdot k+2)\cdot(2\cdot k+3)\cdot(2\cdot k+4)}=\frac{1}{6}\cdot\lim_{n\to\infty}\sum_{k=0}^{n}\int_0^1[(1-x)^3\cdot x^{2\cdot k}]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=0}^{n}[(1-x)^3\cdot x^{2\cdot k}]\cdot dx=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot\sum_{k=0}^{n}x^{2\cdot k}\right]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot\frac{1-x^{2\cdot(n+1)}}{1-x^2}\right]\cdot dx=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(1-x^{2\cdot(n+1)})}{1+x}\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1\frac{(1-x)^2}{1+x}\cdot dx-\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot x^{2\cdot(n+1)}}{1+x}\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1\left(x-3+\frac{4}{x+1}\right)\cdot dx=\frac{1}{6}\cdot\left(\frac{x^2}{2}-3\cdot x+4\cdot\ln(x+1)\right)\Bigg|_0^1$$

$$=\frac{1}{6}\cdot\left(\frac{1}{2}-3+4\cdot\ln 2\right)=\frac{1}{6}\cdot\left(4\cdot\ln 2-\frac{5}{2}\right),$$

since:

$$\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot x^{2\cdot(n+1)}}{1+x}\cdot dx=0.$$

***Example F1.3*: The following equality holds:**

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(3\cdot k+1)\cdot(3\cdot k+2)\cdot(3\cdot k+3)\cdot(3\cdot k+4)}=\frac{1}{6}\cdot\left(1-\frac{3}{2}\cdot\ln 3+\frac{\sqrt{3}\cdot\pi}{6}\right). \tag{F1.3}$$

***Proof*: Equality (F1.3) is obtained from equality (F1), for:**

q=3,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{(3\cdot k+1)\cdot(3\cdot k+2)\cdot(3\cdot k+3)\cdot(3\cdot k+4)}=\frac{1}{6}\cdot\left(\frac{1}{3\cdot k+1}-\frac{3}{3\cdot k+2}+\frac{3}{3\cdot k+3}-\frac{1}{3\cdot k+4}\right)$$

$$=\frac{1}{6}\cdot\int_0^1(x^{3\cdot k}-3\cdot x^{3\cdot k+1}+3\cdot x^{3\cdot k+2}-x^{3\cdot k+3})\cdot dx=\frac{1}{6}\cdot\int_0^1[(1-3\cdot x+3\cdot x^2-x^3)\cdot x^{3\cdot k}]\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1[(1-x)^3\cdot x^{3\cdot k}]\cdot dx.$$

So,

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(3\cdot k+1)\cdot(3\cdot k+2)\cdot(3\cdot k+3)\cdot(3\cdot k+4)}=\frac{1}{6}\cdot\lim_{n\to\infty}\sum_{k=0}^{n}\int_0^1[(1-x)^3\cdot x^{3\cdot k}]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=0}^{n}[(1-x)^3\cdot x^{3\cdot k}]\cdot dx=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot\sum_{k=0}^{n}x^{3\cdot k}\right]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot\frac{1-x^{3\cdot(n+1)}}{1-x^3}\right]\cdot dx=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(1-x^{3\cdot(n+1)})}{1+x+x^2}\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1\frac{(1-x)^2}{1+x+x^2}\cdot dx-\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot x^{3\cdot(n+1)}}{1+x+x^2}\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1\left(1-\frac{3}{2}\cdot\frac{2\cdot x+1}{x^2+x+1}+\frac{3}{2}\cdot\frac{1}{x^2+x+1}\right)\cdot dx$$

$$=\frac{1}{6}\cdot\left(x-\frac{3}{2}\cdot\ln(x^2+x+1)+\sqrt{3}\cdot\operatorname{arctg}\left(\frac{2\cdot x+1}{\sqrt{3}}\right)\right)\Bigg|_0^1=\frac{1}{6}\cdot\left(1-\frac{3}{2}\cdot\ln 3+\frac{\sqrt{3}\cdot\pi}{6}\right),$$

since:

$$\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot x^{3\cdot(n+1)}}{1+x+x^2}\cdot dx=0.$$

Example F1.4: The following equality holds:

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(4\cdot k+1)\cdot(4\cdot k+2)\cdot(4\cdot k+3)\cdot(4\cdot k+4)}=\frac{1}{12}\cdot\left(\frac{3}{2}\cdot\ln 2-\frac{\pi}{2}\right).\qquad\text{(F1.4)}$$

Proof: Equality (F1.4) is obtained from equality (F1), for:

q=4,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{(4\cdot k+1)\cdot(4\cdot k+2)\cdot(4\cdot k+3)\cdot(4\cdot k+4)}=\frac{1}{6}\cdot\left(\frac{1}{4\cdot k+1}-\frac{3}{4\cdot k+2}+\frac{3}{4\cdot k+3}-\frac{1}{4\cdot k+4}\right)$$

$$=\frac{1}{6}\cdot\int_0^1(x^{4\cdot k}-3\cdot x^{4k+1}+3\cdot x^{4\cdot k+2}-x^{4\cdot k+3})\cdot dx=\frac{1}{6}\cdot\int_0^1[(1-3\cdot x+3\cdot x^2-x^3)\cdot x^{4\cdot k}]\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1[(1-x)^3\cdot x^{4\cdot k}]\cdot dx.$$

So,

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(4\cdot k+1)\cdot(4\cdot k+2)\cdot(4\cdot k+3)\cdot(4\cdot k+4)}=\frac{1}{6}\cdot\lim_{n\to\infty}\sum_{k=0}^{n}\int_0^1[(1-x)^3\cdot x^{4\cdot k}]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=0}^{n}[(1-x)^3\cdot x^{4\cdot k}]\cdot dx=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot\sum_{k=0}^{n}x^{4\cdot k}\right]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot\frac{1-x^{4\cdot(n+1)}}{1-x^4}\right]\cdot dx=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot(1-x^{4\cdot(n+1)})}{1+x+x^2+x^3}\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1\frac{(1-x)^2}{1+x+x^2+x^3}\cdot dx-\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot x^{4\cdot(n+1)}}{1+x+x^2+x^3}\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1\left(2\cdot\frac{1}{x+1}-\frac{1}{2}\cdot\frac{2\cdot x}{x^2+1}-\frac{1}{x^2+1}\right)\cdot dx$$

$$=\frac{1}{6}\cdot\left(2\cdot\ln(x+1)-\frac{1}{2}\cdot\ln(x^2+1)-\text{arctg}x\right)\Big|_0^1$$

$$=\frac{1}{6}\cdot\left(2\cdot\ln 2-\frac{1}{2}\cdot\ln 2-\frac{\pi}{4}\right)=\frac{1}{12}\cdot\left(\frac{3}{2}\cdot\ln 2-\frac{\pi}{2}\right),$$

since:

$$\lim_{n\to\infty}\int_0^1\frac{(1-x)^2\cdot x^{4\cdot(n+1)}}{1+x+x^2+x^3}\cdot dx=0.$$

Another category of sequences follows.

G) ***Theorem G***: *If $p, q, r, s, t\in\mathbf{N}^*$ and $p<r<s<t$, then*:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(q\cdot k)\cdot(q\cdot k+p)\cdot(q\cdot k+r)\cdot(q\cdot k+s)\cdot(q\cdot k+t)}=\alpha\cdot\lim_{n\to\infty}\int_0^1\left[\frac{G(x)}{1-x^q}\right]\cdot dx, \quad (G)$$

where:

$G(x)=(\alpha_1-\alpha_2\cdot x^p+\alpha_3\cdot x^r-\alpha_4\cdot x^s+\alpha_5\cdot x^t)\cdot x^{q-1}$,

$$\alpha=\frac{1}{p\cdot r\cdot s\cdot t\cdot(r-p)\cdot(s-p)\cdot(t-p)\cdot(s-r)\cdot(t-r)\cdot(t-s)},$$

$\alpha_1=(r-p)\cdot(s-p)\cdot(t-p)\cdot(s-r)\cdot(t-r)\cdot(t-s)$, $\alpha_2=r\cdot s\cdot t(s-r)\cdot(t-r)\cdot(t-s)$,
$\alpha_3=p\cdot s\cdot t\cdot(s-p)\cdot(t-p)\cdot(t-s)$, $\alpha_4=p\cdot r\cdot t\cdot(r-p)\cdot(t-p)\cdot(t-s)$,
$\alpha_5=p\cdot r\cdot s\cdot(r-p)\cdot(s-p)\cdot(s-r)$.

Proof: For every $k\in\mathbf{N}^*$,

$$\frac{1}{q\cdot k\cdot(q\cdot k+p)\cdot(q\cdot k+r)\cdot(q\cdot k+s)\cdot(q\cdot k+t)}=\alpha\cdot\left(\frac{\alpha_1}{q\cdot k}-\frac{\alpha_2}{q\cdot k+p}+\frac{\alpha_3}{q\cdot k+r}-\frac{\alpha_4}{q\cdot k+s}+\frac{\alpha_5}{q\cdot k+t}\right)$$

$$=\alpha\cdot\int_0^1(\alpha_1\cdot x^{q\cdot k-1}-\alpha_2\cdot x^{q\cdot k+p-1}+\alpha_3\cdot x^{q\cdot k+r-1}-\alpha_4\cdot x^{q\cdot k+s-1}+\alpha_5\cdot x^{q\cdot k+t-1})\cdot dx$$

$$=\alpha\cdot\int_0^1[(\alpha_1-\alpha_2\cdot x^p+\alpha_3\cdot x^r-\alpha_4\cdot x^s+\alpha_5\cdot x^t)\cdot x^{q\cdot k-1}]\cdot dx,$$

where:

$$\alpha=\frac{1}{p\cdot r\cdot s\cdot t\cdot(r-p)\cdot(s-p)\cdot(t-p)\cdot(s-r)\cdot(t-r)\cdot(t-s)},$$

$\alpha_1=(r-p)\cdot(s-p)\cdot(t-p)\cdot(s-r)\cdot(t-r)\cdot(t-s)$, $\alpha_2=r\cdot s\cdot\cdot t(s-r)\cdot(t-r)\cdot(t-s)$,
$\alpha_3=p\cdot s\cdot t\cdot(s-p)\cdot(t-p)\cdot(t-s)$, $\alpha_4=p\cdot r\cdot t\cdot(r-p)\cdot(t-p)\cdot(t-r)$,
$\alpha_5=p\cdot r\cdot s\cdot(r-p)\cdot(s-p)\cdot(s-r)$.

So,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(q\cdot k)\cdot(q\cdot k+p)\cdot(q\cdot k+r)\cdot(q\cdot k+s)\cdot(q\cdot k+t)}=$$

$$=\alpha\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1[(\alpha_1-\alpha_2\cdot x^p+\alpha_3\cdot x^r-\alpha_4\cdot x^s+\alpha_5\cdot x^t)\cdot x^{q\cdot k-1}]\cdot dx$$

$$=\alpha\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}[(\alpha_1-\alpha_2\cdot x^p+\alpha_3\cdot x^r-\alpha_4\cdot x^s+\alpha_5\cdot x^t)\cdot x^{q\cdot k-1}]\cdot dx$$

$$=\alpha\cdot\lim_{n\to\infty}\int_0^1\left[(\alpha_1-\alpha_2\cdot x^p+\alpha_3\cdot x^r-\alpha_4\cdot x^s+\alpha_5\cdot x^t)\cdot\sum_{k=1}^{n}x^{q\cdot k-1}\right]\cdot dx$$

$$=\alpha\cdot\lim_{n\to\infty}\int_0^1\left[(\alpha_1-\alpha_2\cdot x^p+\alpha_3\cdot x^r-\alpha_4\cdot x^s+\alpha_5\cdot x^t)\cdot x^{q-1}\cdot\frac{1-x^{n\cdot q}}{1-x^q}\right]\cdot dx$$

$$=\alpha\cdot\lim_{n\to\infty}\int_0^1\left[\frac{(\alpha_1-\alpha_2\cdot x^p+\alpha_3\cdot x^r-\alpha_4\cdot x^s+\alpha_5\cdot x^t)\cdot x^{q-1}}{1-x^q}\right]\cdot dx$$

$$-\alpha\cdot\lim_{n\to\infty}\int_0^1\left[\frac{(\alpha_1-\alpha_2\cdot x^p+\alpha_3\cdot x^r-\alpha_4\cdot x^s+\alpha_5\cdot x^t)\cdot x^{(n+1)\cdot q-1}}{1-x^q}\right]\cdot dx$$

$$=\alpha\cdot\int_0^1\left[\frac{(\alpha_1-\alpha_2\cdot x^p+\alpha_3\cdot x^r-\alpha_4\cdot x^s+\alpha_5\cdot x^t)\cdot x^{q-1}}{1-x^q}\right]\cdot dx,$$

because, according to (Vălcan, 2016, Proposition),

$$\lim_{n\to\infty}\int_0^1\left[\frac{(\alpha_1-\alpha_2\cdot x^p+\alpha_3\cdot x^r-\alpha_4\cdot x^s+\alpha_5\cdot x^t)\cdot x^{(n+1)\cdot q-1}}{1-x^q}\right]\cdot dx=0.$$

The following remark is required here:

Remark G0: *A simple calculation shows us that*:

$\alpha_1-\alpha_2+\alpha_3-\alpha_4+\alpha_5=0$,

which tells us that none of the above integrals is not improper.

Corollary G1: *If* $q\in\mathbf{N}^*$, *then*:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(q\cdot k)\cdot(q\cdot k+1)\cdot(q\cdot k+2)\cdot(q\cdot k+3)\cdot(q\cdot k+4)}=\frac{1}{24}\cdot\int_0^1\left[\frac{x^{q-1}\cdot(1-x)^4}{1-x^q}\right]\cdot dx. \quad (G1)$$

Proof: Equality (G1) is obtained from Theorem (G), for:

p=1, r=2, s=3 and t=5

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{q\cdot k\cdot(q\cdot k+1)\cdot(q\cdot k+2)\cdot(q\cdot k+3)}=\frac{1}{24}\cdot\left(\frac{1}{q\cdot k}-\frac{4}{q\cdot k+1}+\frac{6}{q\cdot k+2}-\frac{4}{q\cdot k+3}+\frac{1}{q\cdot k+4}\right)$$

$$=\frac{1}{24}\cdot\int_0^1(x^{q\cdot k-1}-4\cdot x^{q\cdot k}+6\cdot x^{q\cdot k+1}-4\cdot x^{q\cdot k+2}+x^{q\cdot k+3})\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1[(1-4\cdot x+6\cdot x^2-4\cdot x^3+x^4)\cdot x^{q\cdot k-1}]\cdot dx=\frac{1}{24}\cdot\int_0^1[(1-x)^4\cdot x^{q\cdot k-1}]\cdot dx.$$

So,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(q\cdot k)\cdot(q\cdot k+1)\cdot(q\cdot k+2)\cdot(q\cdot k+3)\cdot(q\cdot k+4)}=\frac{1}{24}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1[(1-x)^4\cdot x^{q\cdot k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}[(1-x)^4\cdot x^{q\cdot k-1}]\cdot dx=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^4\cdot\sum_{k=1}^{n}x^{q\cdot k-1}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^4\cdot x^{q-1}\cdot\frac{1-x^{n\cdot q}}{1-x^q}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[\frac{x^{q-1}\cdot(1-x)^4}{1-x^q}\right]\cdot dx-\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[\frac{x^{(n+1)\cdot q-1}\cdot(1-x)^4}{1-x^q}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\left[\frac{x^{q-1}\cdot(1-x)^4}{1-x^q}\right]\cdot dx,$$

because, according to (Vălcan, 2016, Proposition),

$$\lim_{n\to\infty}\int_0^1\left[\frac{x^{(n+1)\cdot q-1}\cdot(1-x)^4}{1-x^q}\right]\cdot dx=0.$$

Example G1.1: The following equality holds:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k\cdot(k+1)\cdot(k+2)\cdot(k+3)\cdot(k+4)}=\frac{1}{96}. \quad (G1.1)$$

***Proof*:** Equality (G1.1) is obtained from equality (G1), for:

q=1,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{k\cdot(k+1)\cdot(k+2)\cdot(k+3)\cdot(k+4)}=\frac{1}{24}\cdot\left(\frac{1}{k}-\frac{4}{k+1}+\frac{6}{k+2}-\frac{4}{k+3}+\frac{1}{k+4}\right)$$

$$=\frac{1}{24}\cdot\int_0^1(x^{k-1}-4\cdot x^k+6\cdot x^{k+1}-4\cdot x^{k+2}+x^{k+3})\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1[(1-4\cdot x+6\cdot x^2-4\cdot x^3+x^4)\cdot x^{k-1}]\cdot dx=\frac{1}{24}\cdot\int_0^1[(1-x)^4\cdot x^{k-1}]\cdot dx\,.$$

So,

$$\sum_{k=1}^{n}\frac{1}{k\cdot(k+1)\cdot(k+2)\cdot(k+3)\cdot(k+4)}=\frac{1}{24}\cdot\sum_{k=1}^{n}\int_0^1[(1-x)^4\cdot x^{k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\sum_{k=1}^{n}[(1-x)^4\cdot x^{k-1}]\cdot dx=\frac{1}{24}\cdot\int_0^1\left[(1-x)^4\cdot\sum_{k=1}^{n}x^{k-1}\right]\cdot dx=\frac{1}{24}\cdot\int_0^1\left[(1-x)^4\cdot\frac{1-x^n}{1-x}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1[(1-x)^3\cdot(1-x^n)]\cdot dx\,.$$

It follows that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k\cdot(k+1)\cdot(k+2)\cdot(k+3)\cdot(k+4)}=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1[(1-x)^3\cdot(1-x^n)]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1(1-x)^3\cdot dx-\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1[(1-x)^3\cdot x^n]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1(1-3\cdot x+3\cdot x^2-x^3)\cdot dx=\frac{1}{24}\cdot\left(x-3\cdot\frac{x^2}{2}+3\cdot\frac{x^3}{3}-\frac{x^4}{4}\right)\Bigg|_0^1$$

$$=\frac{1}{24}\cdot\left(1-\frac{3}{2}+\frac{3}{3}-\frac{1}{4}\right)=\frac{1}{24}\cdot\frac{1}{4}=\frac{1}{96}\,,$$

since:

$$\lim_{n\to\infty}\int_0^1[(1-x)^3\cdot x^n]\cdot dx=0.$$

***Example G1.2*:** The following equality holds:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k+1)\cdot(2\cdot k+2)\cdot(2\cdot k+3)\cdot(2\cdot k+4)}=\frac{1}{24}\cdot\left(\frac{67}{12}-8\cdot\ln 2\right).\qquad\text{(G1.2)}$$

***Proof*:** Equality (G1.2) is obtained from equality (G1), for:

q=2,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{(2\cdot k)\cdot(2\cdot k+1)\cdot(2\cdot k+2)\cdot(2\cdot k+3)\cdot(2\cdot k+4)}$$

$$=\frac{1}{24}\cdot\left(\frac{1}{2\cdot k}-\frac{4}{2\cdot k+1}+\frac{6}{2\cdot k+2}-\frac{4}{2\cdot k+3}+\frac{1}{2\cdot k+4}\right)$$

$$=\frac{1}{24}\cdot\int_0^1(x^{2\cdot k-1}-4\cdot x^{2\cdot k}+6\cdot x^{2\cdot k+1}-4\cdot x^{2\cdot k+2}+x^{2\cdot k+3})\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1[(1-4\cdot x+6\cdot x^2-4\cdot x^3+x^4)\cdot x^{2\cdot k-1}]\cdot dx=\frac{1}{24}\cdot\int_0^1[(1-x)^4\cdot x^{2\cdot k-1}]\cdot dx\,.$$

So,

$$\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k+1)\cdot(2\cdot k+2)\cdot(2\cdot k+3)\cdot(2\cdot k+4)}=\frac{1}{24}\cdot\sum_{k=1}^{n}\int_0^1[(1-x)^4\cdot x^{2\cdot k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\sum_{k=1}^{n}[(1-x)^4\cdot x^{2\cdot k-1}]\cdot dx=\frac{1}{24}\cdot\int_0^1\left[(1-x)^4\cdot\sum_{k=1}^{n}x^{2\cdot k-1}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\left[(1-x)^4\cdot x\cdot\frac{1-x^{2\cdot n}}{1-x^2}\right]\cdot dx=\frac{1}{24}\cdot\int_0^1\left[(1-x)^3\cdot x\cdot\frac{1-x^{2\cdot n}}{1+x}\right]\cdot dx.$$

It follows that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k+1)\cdot(2\cdot k+2)\cdot(2\cdot k+3)\cdot(2\cdot k+4)}$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot x\cdot\frac{1-x^{2\cdot n}}{1+x}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\left[\frac{(1-x)^3\cdot x}{1+x}\right]\cdot dx-\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot\frac{x^{2\cdot n+1}}{1+x}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\frac{x-3\cdot x^2+3\cdot x^3-x^4}{1+x}\cdot dx=\frac{1}{24}\cdot\int_0^1\left(8-7\cdot x+4\cdot x^2-x^3-8\cdot\frac{1}{1+x}\right)\cdot dx$$

$$=\frac{1}{24}\cdot\left(8\cdot x-7\cdot\frac{x^2}{2}+4\cdot\frac{x^3}{3}-\frac{x^4}{4}-8\cdot\ln(x+1)\right)\Bigg|_0^1$$

$$=\frac{1}{24}\cdot\left(8-\frac{7}{2}+\frac{4}{3}-\frac{1}{4}-8\cdot\ln 2\right)=\frac{1}{24}\cdot\left(\frac{67}{12}-8\cdot\ln 2\right),$$

since:

$$\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot\frac{x^{2\cdot n+1}}{1+x}\right]\cdot dx=0.$$

***Example G1.3*:** The following equality holds:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k+1)\cdot(3\cdot k+2)\cdot(3\cdot k+3)\cdot(3\cdot k+4)}=\frac{1}{48}\cdot\left(\frac{13}{6}+3\cdot\ln 3-\pi\cdot\sqrt{3}\right). \qquad \text{(G1.3)}$$

***Proof*:** Equality (G1.3) is obtained from equality (G1), for:

$q=3$,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{(3\cdot k)\cdot(3\cdot k+1)\cdot(3\cdot k+2)\cdot(3\cdot k+3)\cdot(3\cdot k+4)}=\frac{1}{24}\cdot\left(\frac{1}{3\cdot k}-\frac{4}{3\cdot k+1}+\frac{6}{3\cdot k+2}-\frac{4}{3\cdot k+3}+\frac{1}{3\cdot k+4}\right)$$

$$=\frac{1}{24}\cdot\int_0^1(x^{3\cdot k-1}-4\cdot x^{3\cdot k}+6\cdot x^{3\cdot k+1}-4\cdot x^{3\cdot k+2}+x^{3\cdot k+3})\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1[(1-4\cdot x+6\cdot x^2-4\cdot x^3+x^4)\cdot x^{3\cdot k-1}]\cdot dx=\frac{1}{24}\cdot\int_0^1[(1-x)^4\cdot x^{3\cdot k-1}]\cdot dx.$$

So,

$$\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k+1)\cdot(3\cdot k+2)\cdot(3\cdot k+3)\cdot(3\cdot k+4)}=\frac{1}{24}\cdot\sum_{k=1}^{n}\int_0^1[(1-x)^4\cdot x^{3\cdot k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\sum_{k=1}^{n}[(1-x)^4\cdot x^{3\cdot k-1}]\cdot dx=\frac{1}{24}\cdot\int_0^1\left[(1-x)^4\cdot\sum_{k=1}^{n}x^{3\cdot k-1}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\left[(1-x)^4\cdot x^2\cdot\frac{1-x^{3\cdot n}}{1-x^3}\right]\cdot dx=\frac{1}{24}\cdot\int_0^1\left[(1-x)^3\cdot x^2\cdot\frac{1-x^{3\cdot n}}{1+x+x^2}\right]\cdot dx\,.$$

It follows that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k+1)\cdot(3\cdot k+2)\cdot(3\cdot k+3)\cdot(3\cdot k+4)}=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot x^2\cdot\frac{1-x^{3\cdot n}}{1+x+x^2}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\left[\frac{(1-x)^3\cdot x^2}{1+x+x^2}\right]\cdot dx-\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot\frac{x^{3\cdot n+2}}{1+x+x^2}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\frac{x^2-3\cdot x^3+3\cdot x^4-x^5}{1+x+x^2}\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\left(3-6\cdot x+4\cdot x^2-x^3+\frac{3}{2}\cdot\frac{2\cdot x+1}{x^2+x+1}-\frac{9}{2}\cdot\frac{1}{x^2+x+1}\right)\cdot dx$$

$$=\frac{1}{24}\cdot\left(3\cdot x-6\cdot\frac{x^2}{2}+4\cdot\frac{x^3}{3}-\frac{x^4}{4}+\frac{3}{2}\cdot\ln(x^2+x+1)-3\cdot\sqrt{3}\cdot\operatorname{arctg}\left(\frac{2\cdot x+1}{\sqrt{3}}\right)\right)\Bigg|_0^1$$

$$=\frac{1}{24}\cdot\left(3-\frac{6}{2}+\frac{4}{3}-\frac{1}{4}+\frac{3}{2}\cdot\ln3-3\cdot\sqrt{3}\cdot\frac{\pi}{6}\right)\Bigg|_0^1$$

$$=\frac{1}{24}\cdot\left(\frac{13}{12}+\frac{3}{2}\cdot\ln3-\frac{\pi\cdot\sqrt{3}}{2}\right)=\frac{1}{48}\cdot\left(\frac{13}{6}+3\cdot\ln3-\pi\cdot\sqrt{3}\right),$$

since:

$$\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot\frac{x^{3\cdot n+2}}{1+x+x^2}\right]\cdot dx=0.$$

Example G1.4: The following equality holds:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(4\cdot k)\cdot(4\cdot k+1)\cdot(4\cdot k+2)\cdot(4\cdot k+3)\cdot(4\cdot k+4)}=\frac{1}{24}\cdot\left(\frac{25}{12}-3\cdot\ln2\right).\qquad(G1.4)$$

Proof: Equality (G1.4) is obtained from equality (G1), for:

q=4,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{(4\cdot k)\cdot(4\cdot k+1)\cdot(4\cdot k+2)\cdot(4\cdot k+3)\cdot(4\cdot k+4)}=\frac{1}{24}\cdot\left(\frac{1}{4\cdot k}-\frac{4}{4\cdot k+1}+\frac{6}{4\cdot k+2}-\frac{4}{4\cdot k+3}+\frac{1}{4\cdot k+4}\right)$$

$$=\frac{1}{24}\cdot\int_0^1(x^{4\cdot k-1}-4\cdot x^{4\cdot k}+6\cdot x^{4\cdot k+1}-4\cdot x^{4\cdot k+2}+x^{4\cdot k+3})\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1[(1-4\cdot x+6\cdot x^2-4\cdot x^3+x^4)\cdot x^{4\cdot k-1}]\cdot dx=\frac{1}{24}\cdot\int_0^1[(1-x)^4\cdot x^{4\cdot k-1}]\cdot dx\,.$$

So,

$$\sum_{k=1}^{n}\frac{1}{(4\cdot k)\cdot(4\cdot k+1)\cdot(4\cdot k+2)\cdot(4\cdot k+3)\cdot(4\cdot k+4)}=\frac{1}{24}\cdot\sum_{k=1}^{n}\int_0^1[(1-x)^4\cdot x^{4\cdot k-1}]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\sum_{k=1}^{n}[(1-x)^4\cdot x^{4\cdot k-1}]\cdot dx=\frac{1}{24}\cdot\int_0^1\left[(1-x)^4\cdot\sum_{k=1}^{n}x^{4\cdot k-1}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\left[(1-x)^4\cdot x^3\cdot\frac{1-x^{4\cdot n}}{1-x^4}\right]\cdot dx=\frac{1}{24}\cdot\int_0^1\left[(1-x)^3\cdot x^3\cdot\frac{1-x^{4\cdot n}}{1+x+x^2+x^3}\right]\cdot dx\,.$$

It follows that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(4\cdot k)\cdot(4\cdot k+1)\cdot(4\cdot k+2)\cdot(4\cdot k+3)\cdot(4\cdot k+4)}$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot x^3\cdot\frac{1-x^{4\cdot n}}{1+x+x^2+x^3}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\left[\frac{(1-x)^3\cdot x^3}{1+x+x^2+x^3}\right]\cdot dx-\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot\frac{x^{4\cdot n+3}}{1+x+x^2+x^3}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\frac{x^3-3\cdot x^4+3\cdot x^5-x^6}{1+x+x^2+x^3}\cdot dx=\frac{1}{24}\cdot\int_0^1\left(4-6\cdot x+4\cdot x^2-x^3-4\cdot\frac{1}{x+1}+\frac{2\cdot x}{x^2+1}\right)\cdot dx$$

$$=\frac{1}{24}\cdot\left(4\cdot x-6\cdot\frac{x^2}{2}+4\cdot\frac{x^3}{3}-\frac{x^4}{4}-4\cdot\ln(x+1)+\ln(x^2+1)\right)\Bigg|_0^1$$

$$=\frac{1}{24}\cdot\left(4-\frac{6}{2}+\frac{4}{3}-\frac{1}{4}-4\cdot\ln 2+\ln 2\right)\Bigg|_0^1=\frac{1}{24}\cdot\left(\frac{25}{12}-3\cdot\ln 2\right),$$

since:

$$\lim_{n\to\infty}\int_0^1\left[(1-x)^3\cdot\frac{x^{4\cdot n+3}}{1+x+x^2+x^3}\right]\cdot dx=0.$$

By reasoning inductively, we immediately arrive at the next result, which we can obtain and by reasoning deductively, from (Vălcan, 2016, Equalities (10)).

***Example G1.p*:** For every p, $q\in\mathbf{N}^*$, the following equality holds:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(q\cdot k)\cdot(q\cdot k+1)\cdot(q\cdot k+2)\cdot\dots\cdot(q\cdot k+p)}=\frac{1}{p!}\cdot\int_0^1\left[\frac{x^{q-1}\cdot(1-x)^p}{1-x^q}\right]\cdot dx\,. \tag{G1.p}$$

particularly,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k\cdot(k+1)\cdot(k+2)\cdot\dots\cdot(k+p)}=\frac{1}{p!}\cdot\int_0^1(1-x)^{p-1}\cdot dx$$

$$=\frac{1}{p!}\cdot\left(C_{p-1}^0-\frac{1}{2}\cdot C_{p-1}^1+\frac{1}{3}\cdot C_{p-1}^2-\dots+(-1)^{p-2}\cdot\frac{1}{p-1}\cdot C_{p-1}^{p-2}+(-1)^{p-1}\cdot\frac{1}{p}\cdot C_{p-1}^{p-1}\right). \tag{G1.p′}$$

Finally, we have reached the last category of sequences whose limit we want to calculate.

H) *Theorem H*: *If p, q, r, s, t, u* $\in\mathbf{N}^*$ *and* $p<r<s<t<u$, *then*:

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(q\cdot k+p)\cdot(q\cdot k+r)\cdot(q\cdot k+s)\cdot(q\cdot k+t)\cdot(q\cdot k+u)}=\alpha\cdot\lim_{n\to\infty}\int_0^1\left[\frac{H(x)}{1-x^q}\right]\cdot dx\,, \tag{H}$$

where:

$H(x)=(\alpha_1-\alpha_2\cdot x^{r-p}+\alpha_3\cdot x^{s-p}-\alpha_4\cdot x^{t-p}+\alpha_5\cdot x^{u-p})\cdot x^{p-1}$,

$$\alpha=\frac{1}{(r-p)\cdot(s-p)\cdot(t-p)\cdot(u-p)\cdot(s-r)\cdot(t-r)\cdot(u-r)\cdot(t-s)\cdot(u-s)\cdot(t-s)},$$

$\alpha_1=(s-r)\cdot(t-r)\cdot(u-r)\cdot(t-s)\cdot(u-s)\cdot(u-t)$, $\alpha_2=(s-p)\cdot(t-p)\cdot(u-p)\cdot(t-s)\cdot(u-s)\cdot(u-t)$,

$\alpha_3=(r-p)\cdot(t-p)\cdot(u-p)\cdot(t-r)\cdot(u-r)\cdot(u-t)$, $\alpha_4=(r-p)\cdot(s-p)\cdot(u-p)\cdot(s-s)\cdot(u-s)\cdot(u-s)$,

$\alpha_5=(r-p)\cdot(s-p)\cdot(t-p)\cdot(s-r)\cdot(t-r)\cdot(t-s)$.

***Proof*:** For every $k\in\mathbf{N}^*$,

$$\frac{1}{(q\cdot k+p)\cdot(q\cdot k+r)\cdot(q\cdot k+s)\cdot(q\cdot k+t)\cdot(q\cdot k+u)}=$$

$$=\alpha\cdot\left(\frac{\alpha_1}{q\cdot k+p}-\frac{\alpha_2}{q\cdot k+r}+\frac{\alpha_3}{q\cdot k+s}-\frac{\alpha_4}{q\cdot k+t}+\frac{\alpha_5}{q\cdot k+u}\right)$$

$$=\alpha\cdot\int_0^1(\alpha_1\cdot x^{q\cdot k+p-1}-\alpha_2\cdot x^{q\cdot k+r-1}+\alpha_3\cdot x^{q\cdot k+s-1}-\alpha_4\cdot x^{q\cdot k+t-1}+\alpha_5\cdot x^{q\cdot k+u-1})\cdot dx$$

$$=\alpha\cdot\int_0^1[(\alpha_1-\alpha_2\cdot x^{r-p}+\alpha_3\cdot x^{s-p}-\alpha_4\cdot x^{t-p}+\alpha_5\cdot x^{u-p})\cdot x^{q\cdot k+p-1}]\cdot dx\,,$$

where:

$$\alpha=\frac{1}{(r-p)\cdot(s-p)\cdot(t-p)\cdot(u-p)\cdot(s-r)\cdot(t-r)\cdot(u-r)\cdot(t-s)\cdot(u-s)\cdot(t-s)}\,,$$

$\alpha_1=(s\text{-}r)\cdot(t\text{-}r)\cdot(u\text{-}r)\cdot(t\text{-}s)\cdot(u\text{-}s)\cdot(u\text{-}t)$, $\alpha_2=(s\text{-}p)\cdot(t\text{-}p)\cdot(u\text{-}p)\cdot(t\text{-}s)\cdot(u\text{-}s)\cdot(u\text{-}t)$,
$\alpha_3=(r\text{-}p)\cdot(t\text{-}p)\cdot(u\text{-}p)\cdot(t\text{-}r)\cdot(u\text{-}r)\cdot(u\text{-}t)$, $\alpha_4=(r\text{-}p)\cdot(s\text{-}p)\cdot(u\text{-}p)\cdot(s\text{-}s)\cdot(u\text{-}s)\cdot(u\text{-}s)$,
$\alpha_5=(r\text{-}p)\cdot(s\text{-}p)\cdot(t\text{-}p)\cdot(s\text{-}r)\cdot(t\text{-}r)\cdot(t\text{-}s)$.

So,

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(q\cdot k+p)\cdot(q\cdot k+r)\cdot(q\cdot k+s)\cdot(q\cdot k+t)\cdot(q\cdot k+u)}=$$

$$=\alpha\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1[(\alpha_1-\alpha_2\cdot x^{r-p}+\alpha_3\cdot x^{s-p}-\alpha_4\cdot x^{t-p}+\alpha_5\cdot x^{u-p})\cdot x^{q\cdot k+p-1}]\cdot dx$$

$$=\alpha\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}[(\alpha_1-\alpha_2\cdot x^{r-p}+\alpha_3\cdot x^{s-p}-\alpha_4\cdot x^{t-p}+\alpha_5\cdot x^{u-p})\cdot x^{q\cdot k+p-1}]\cdot dx$$

$$=\alpha\cdot\lim_{n\to\infty}\int_0^1\left[(\alpha_1-\alpha_2\cdot x^{r-p}+\alpha_3\cdot x^{s-p}-\alpha_4\cdot x^{t-p}+\alpha_5\cdot x^{u-p})\cdot x^{p-1}\cdot\sum_{k=1}^{n}x^{q\cdot k}\right]\cdot dx$$

$$=\alpha\cdot\lim_{n\to\infty}\int_0^1\left[(\alpha_1-\alpha_2\cdot x^{r-p}+\alpha_3\cdot x^{s-p}-\alpha_4\cdot x^{t-p}+\alpha_5\cdot x^{u-p})\cdot x^{p-1}\cdot\frac{1-x^{(n+1)\cdot q}}{1-x^q}\right]\cdot dx$$

$$=\alpha\cdot\lim_{n\to\infty}\int_0^1\left[\frac{(\alpha_1-\alpha_2\cdot x^{r-p}+\alpha_3\cdot x^{s-p}-\alpha_4\cdot x^{t-p}+\alpha_5\cdot x^{u-p})\cdot x^{p-1}}{1-x^q}\right]\cdot dx$$

$$-\alpha\cdot\lim_{n\to\infty}\int_0^1\left[\frac{(\alpha_1-\alpha_2\cdot x^{r-p}+\alpha_3\cdot x^{s-r}-\alpha_4\cdot x^{t-p}+\alpha_5\cdot x^{u-p})\cdot x^{p-1}\cdot x^{(n+1)\cdot q}}{1-x^q}\right]\cdot dx$$

$$=\alpha\cdot\int_0^1\left[\frac{(\alpha_1-\alpha_2\cdot x^{r-p}+\alpha_3\cdot x^{s-p}-\alpha_4\cdot x^{t-p}+\alpha_5\cdot x^{u-p})\cdot x^{p-1}}{1-x^q}\right]\cdot dx\,,$$

because, according to (Vălcan, 2016, Proposition),

$$\lim_{n\to\infty}\int_0^1\left[\frac{(\alpha_1-\alpha_2\cdot x^{r-p}+\alpha_3\cdot x^{s-r}-\alpha_4\cdot x^{t-p}+\alpha_5\cdot x^{u-p})\cdot x^{p-1}\cdot x^{(n+1)\cdot q}}{1-x^q}\right]\cdot dx=0.$$

The following remark is required here:

***Remark H0**: A simple calculation shows us that*:

$\alpha_1-\alpha_2+\alpha_3-\alpha_4+\alpha_5=0$,

which tells us that none of the above integrals is not improper.

***Corollary H1:** If $q\in\mathbf{N}^*$, then*:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(q\cdot k+1)\cdot(q\cdot k+2)\cdot(q\cdot k+3)\cdot(q\cdot k+4)\cdot(q\cdot k+5)}=\frac{1}{24}\cdot\int_0^1\frac{(1-x)^4}{1-x^q}\cdot dx\,. \qquad \text{(H1)}$$

Proof: Equality (H1) is obtained from Theorem (H), for:

p=1, r=2, s=3, t=4 and u=5

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{(q\cdot k+1)\cdot(q\cdot k+2)\cdot(q\cdot k+3)\cdot(q\cdot k+4)\cdot(q\cdot k+5)}=$$

$$=\frac{1}{24}\cdot\left(\frac{1}{q\cdot k+1}-\frac{4}{q\cdot k+2}+\frac{6}{q\cdot k+3}-\frac{4}{q\cdot k+4}+\frac{1}{q\cdot k+5}\right)$$

$$=\frac{1}{24}\cdot\int_0^1 (x^{q\cdot k}-4\cdot x^{q\cdot k+1}+6\cdot x^{q\cdot k+2}-4\cdot x^{q\cdot k+3}+x^{q\cdot k+4})\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1 [(1-4\cdot x+6\cdot x^2-4\cdot x^3+x^4)\cdot x^{q\cdot k}]\cdot dx=\frac{1}{24}\cdot\int_0^1 [(1-x)^4\cdot x^{q\cdot k}]\cdot dx\,.$$

So,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(q\cdot k+1)\cdot(q\cdot k+)\cdot(q\cdot k+3)\cdot(q\cdot k+4)\cdot(q\cdot k+5)}=\frac{1}{24}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1 [(1-x)^4\cdot x^{q\cdot k}]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}[(1-x)^4\cdot x^{q\cdot k}]\cdot dx=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^4\cdot\sum_{k=1}^{n}x^{q\cdot k}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^4\cdot\frac{1-x^{(n+1)\cdot q}}{1-x^q}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^4}{1-x^q}\cdot dx-\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1\left[\frac{x^{(n+1)\cdot q}\cdot(1-x)^4}{1-x^q}\right]\cdot dx=\frac{1}{24}\cdot\int_0^1\frac{(1-x)^4}{1-x^q}\cdot dx\,,$$

because, according to (Vălcan, 2016, Proposition),

$$\lim_{n\to\infty}\int_0^1\left[\frac{x^{(n+1)\cdot q}\cdot(1-x)^4}{1-x^q}\right]\cdot dx=0.$$

Example H1.1: The following equality holds:

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(k+1)\cdot(k+2)\cdot(k+3)\cdot(k+4)\cdot(k+5)}=\frac{1}{96}\,. \tag{H1.1}$$

Proof: Equality (H1.1) is obtained from equality (H1), for:

q=1,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{(k+1)\cdot(k+2)\cdot(k+3)\cdot(k+4)\cdot(k+5)}=\frac{1}{24}\cdot\left(\frac{1}{k+1}-\frac{4}{k+2}+\frac{6}{k+3}-\frac{4}{k+4}+\frac{1}{k+5}\right)$$

$$=\frac{1}{24}\cdot\int_0^1 (x^{k}-4\cdot x^{k+1}+6\cdot x^{k+2}-4\cdot x^{k+3}+x^{k+4\cdot})\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1 [(1-4\cdot x+6\cdot x^2-4\cdot x^3+x^{4\cdot})\cdot x^{k}]\cdot dx=\frac{1}{24}\cdot\int_0^1 [(1-x)^4\cdot x^{k}]\cdot dx\,.$$

So,

$$\sum_{k=0}^{n}\frac{1}{(k+1)\cdot(k+2)\cdot(k+3)\cdot(k+4)\cdot(k+5)}=\frac{1}{24}\cdot\sum_{k=1}^{n}\int_0^1 [(1-x)^4\cdot x^{k}]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1\sum_{k=1}^{n}[(1-x)^4\cdot x^{k}]\cdot dx=\frac{1}{24}\cdot\int_0^1\left[(1-x)^4\cdot\sum_{k=1}^{n}x^{k}\right]\cdot dx=\frac{1}{24}\cdot\int_0^1\left[(1-x)^4\cdot\frac{1-x^{n+1}}{1-x}\right]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1 [(1-x)^3\cdot(1-x^{n+1})]\cdot dx\,.$$

It follows that:

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(k+1)\cdot(k+2)\cdot(k+3)\cdot(k+4)\cdot(k+5)}=\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1 [(1-x)^3\cdot(1-x^{n+1})]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1 (1-x)^3\cdot dx-\frac{1}{24}\cdot\lim_{n\to\infty}\int_0^1 [(1-x)^3\cdot x^{n+1}]\cdot dx$$

$$=\frac{1}{24}\cdot\int_0^1(1-3\cdot x+3\cdot x^2-x^3)\cdot dx=\frac{1}{24}\cdot\left(x-3\cdot\frac{x^2}{2}+3\cdot\frac{x^3}{3}-\frac{x^4}{4}\right)\Big|_0^1$$

$$=\frac{1}{24}\cdot\left(1-\frac{3}{2}+\frac{3}{3}-\frac{1}{4}\right)=\frac{1}{24}\cdot\frac{1}{4}=\frac{1}{96},$$

since:

$$\lim_{n\to\infty}\int_0^1[(1-x)^3\cdot x^{n+1}]\cdot dx=0.$$

Example H1.2: The following equality holds:

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(2\cdot k+1)\cdot(2\cdot k+2)\cdot(2\cdot k+3)\cdot(2\cdot k+4)\cdot(2\cdot k+5)}=\frac{1}{6}\cdot\left(8\cdot\ln 2-\frac{16}{3}\right). \qquad \text{(H1.2)}$$

Proof: Equality (H1.2) is obtained from equality (H1), for:

q=2,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{(2\cdot k+1)\cdot(2\cdot k+2)\cdot(2\cdot k+3)\cdot(2\cdot k+4)\cdot(2\cdot k+5)}=$$

$$=\frac{1}{6}\cdot\left(\frac{1}{2\cdot k+1}-\frac{4}{2\cdot k+2}+\frac{6}{2\cdot k+3}-\frac{4}{2\cdot k+4}+\frac{1}{2\cdot k+5}\right)$$

$$=\frac{1}{6}\cdot\int_0^1(x^{2\cdot k}-4\cdot x^{2\cdot k+1}+6\cdot x^{2\cdot k+2}-4\cdot x^{2\cdot k+3}+x^{2\cdot k+4})\cdot dx=\frac{1}{6}\cdot\int_0^1[(1-x)^3\cdot x^{2\cdot k}]\cdot dx.$$

So,

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(2\cdot k+1)\cdot(2\cdot k+2)\cdot(2\cdot k+3)\cdot(2\cdot k+4)\cdot(2\cdot k+5)}=\lim_{n\to\infty}\frac{1}{6}\cdot\sum_{k=0}^{n}\int_0^1[(1-x)^4\cdot x^{2\cdot k}]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=0}^{n}[(1-x)^4\cdot x^{2\cdot k}]\cdot dx=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^4\cdot\sum_{k=0}^{n}x^{2\cdot k}\right]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^4\cdot\frac{1-x^{2\cdot n+2}}{1-x^2}\right]\cdot dx=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^3\cdot(1-x^{2\cdot n+2})}{1+x}\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1\frac{(1-x)^3}{1+x}\cdot dx-\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^3\cdot x^{2\cdot n+2}}{1+x}\cdot dx=\frac{1}{6}\cdot\int_0^1\left(-x^2+4\cdot x-7+\frac{8}{x+1}\right)\cdot dx$$

$$=\frac{1}{6}\cdot\left(-\frac{1}{3}+2-7+8\cdot\ln 2\right)=\frac{1}{6}\cdot\left(8\cdot\ln 2-\frac{16}{3}\right),$$

since:

$$\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^3\cdot x^{2\cdot n+2}}{1+x}\cdot dx=0.$$

Example H1.3: The following equality holds:

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(3\cdot k+1)\cdot(3\cdot k+2)\cdot(3\cdot k+3)\cdot(3\cdot k+4)\cdot(3\cdot k+5)}=\frac{1}{6}\cdot\left(\frac{7}{2}-3\cdot\ln 3\right). \qquad \text{(H1.3)}$$

Proof: Equality (H1.3) is obtained from equality (H1), for:

q=3,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{(3\cdot k+1)\cdot(3\cdot k+2)\cdot(3\cdot k+3)\cdot(3\cdot k+4)\cdot(3\cdot k+5)}=$$

$$=\frac{1}{6}\cdot\left(\frac{1}{3\cdot k+1}-\frac{4}{3\cdot k+2}+\frac{6}{3\cdot k+3}-\frac{4}{3\cdot k+4}+\frac{1}{3\cdot k+5}\right)$$

$$=\frac{1}{6}\cdot\int_0^1(x^{3\cdot k}-4\cdot x^{3\cdot k+1}+6\cdot x^{3\cdot k+2}-4\cdot x^{3\cdot k+3}+x^{3\cdot k+4})\cdot dx=\frac{1}{6}\cdot\int_0^1[(1-x)^4\cdot x^{3\cdot k}]\cdot dx.$$

So,

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(3\cdot k+1)\cdot(3\cdot k+2)\cdot(3\cdot k+3)\cdot(3\cdot k+4)\cdot(3\cdot k+5)}=\lim_{n\to\infty}\frac{1}{6}\cdot\sum_{k=0}^{n}\int_0^1[(1-x)^4\cdot x^{3\cdot k}]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=0}^{n}[(1-x)^4\cdot x^{3\cdot k}]\cdot dx=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^4\cdot\sum_{k=0}^{n}x^{3\cdot k}\right]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^4\cdot\frac{1-x^{3\cdot n+3}}{1-x^3}\right]\cdot dx=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^3\cdot(1-x^{3\cdot n+3})}{1+x+x^2}\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1\frac{(1-x)^3}{1+x+x^2}\cdot dx-\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^3\cdot x^{3\cdot n+3}}{1+x+x^2}\cdot dx=\frac{1}{6}\cdot\int_0^1\left(-x+4-3\cdot\frac{2\cdot x+1}{x^2+x+1}\right)\cdot dx$$

$$=\frac{1}{6}\cdot\left(-\frac{1}{2}+4-3\cdot\ln 3\right)=\frac{1}{6}\cdot\left(\frac{7}{2}-3\cdot\ln 3\right),$$

since:

$$\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^3\cdot x^{3\cdot n+3}}{1+x+x^2}\cdot dx=0.$$

***Example H1.4*:** The following equality holds:

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(4\cdot k+1)\cdot(4\cdot k+2)\cdot(4\cdot k+3)\cdot(4\cdot k+4)\cdot(4\cdot k+5)}=\frac{1}{6}\cdot\left(4\cdot\ln 2-\frac{\pi}{2}-1\right).\qquad\text{(H1.4)}$$

***Proof*:** Equality (H1.4) is obtained from equality (H1), for:

q=4,

or by direct calculation. Thus, for every $k\in\mathbf{N}^*$,

$$\frac{1}{(4\cdot k+1)\cdot(4\cdot k+2)\cdot(4\cdot k+3)\cdot(4\cdot k+4)\cdot(4\cdot k+5)}=$$

$$=\frac{1}{6}\cdot\left(\frac{1}{4\cdot k+1}-\frac{4}{4\cdot k+2}+\frac{6}{4\cdot k+3}-\frac{4}{4\cdot k+4}+\frac{1}{4\cdot k+5}\right).$$

$$=\frac{1}{6}\cdot\int_0^1(x^{4\cdot k}-4\cdot x^{4\cdot k+1}+6\cdot x^{4\cdot k+2}-4\cdot x^{4\cdot k+3}+x^{4\cdot k+4})\cdot dx=\frac{1}{6}\cdot\int_0^1[(1-x)^4\cdot x^{4\cdot k}]\cdot dx$$

So,

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(4\cdot k+1)\cdot(4\cdot k+2)\cdot(4\cdot k+3)\cdot(4\cdot k+4)\cdot(4\cdot k+5)}=\lim_{n\to\infty}\frac{1}{6}\cdot\sum_{k=0}^{n}\int_0^1[(1-x)^4\cdot x^{4\cdot k}]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=0}^{n}[(1-x)^4\cdot x^{4\cdot k}]\cdot dx\ \frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^4\cdot\sum_{k=0}^{n}x^{4\cdot k}\right]\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^4\cdot\frac{1-x^{4\cdot n+1}}{1-x^4}\right]\cdot dx=\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^3\cdot(1-x^{4\cdot n+1})}{1+x+x^2+x^3}\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1\frac{(1-x)^3}{1+x+x^2+x^3}\cdot dx-\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^3\cdot x^{4\cdot n+1}}{1+x+x^2+x^3}\cdot dx$$

$$=\frac{1}{6}\cdot\int_0^1\left(-1+4\cdot\frac{1}{x+1}-2\cdot\frac{1}{x^2+1}\right)\cdot dx=\frac{1}{6}\cdot\left(4\cdot\ln 2-\frac{\pi}{2}-1\right),$$

since:

$$\frac{1}{6}\cdot\lim_{n\to\infty}\int_0^1\frac{(1-x)^3\cdot x^{4\cdot n+1}}{1+x+x^2+x^3}\cdot dx=0.$$

By reasoning inductively, we immediately arrive at the next result, which we can obtain and by reasoning deductively, from (Vălcan, 2016, Equalities (10)).

***Example H1.p*:** For every p, q$\in\mathbf{N}^*$, the following equality holds:

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(q\cdot k+1)\cdot(q\cdot k+2)\cdot\dots\cdot(q\cdot k+p+1)}=\frac{1}{p!}\cdot\int_0^1\frac{(1-x)^p}{1-x^q}\cdot dx. \qquad \text{(H1.p)}$$

particularly,

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(k+1)\cdot(k+2)\cdot\dots\cdot(k+p+1)}=\frac{1}{p!}\cdot\int_0^1(1-x)^{p-1}\cdot dx$$

$$=\frac{1}{p!}\cdot\left(C_{p-1}^0-\frac{1}{2}\cdot C_{p-1}^1+\frac{1}{3}\cdot C_{p-1}^2-\dots+(-1)^{p-2}\cdot\frac{1}{p-1}\cdot C_{p-1}^{p-2}+(-1)^{p-1}\cdot\frac{1}{p}\cdot C_{p-1}^{p-1}\right). \qquad \text{(H1.p')}$$

Equality (H1.p′) results from equality (H1.p) or from (Vălcan, 2016, Equalities (10′)) and taking into account the fact that, for every k$\in\mathbf{N}$,

$$\frac{1}{(k+1)\cdot(k+2)\cdot\ldots\cdot(k+p+1)}=\frac{1}{p!}\cdot\int_0^1[(1-x)^p\cdot x^k]\cdot dx$$

$$=\frac{1}{p!}\cdot\int_0^1\left[x^k-C_p^1\cdot x^{k+1}+C_p^2\cdot x^{k+2}-\ldots+(-1)^{p-1}\cdot C_p^{p-1}\cdot x^{k+p-1}+(-1)^p\cdot C_p^p\cdot x^{k+p}\right]$$

$$=\frac{1}{0!\cdot p!}\cdot\frac{1}{k+1}-\frac{1}{1!\cdot(p-1)!}\cdot\frac{1}{k+2}+\ldots+(-1)^p\cdot\frac{1}{p!\cdot 0!}\cdot\frac{1}{k+p+1}.$$

So,

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(k+1)\cdot(k+2)\cdot\dots\cdot(k+p+1)}=\frac{1}{p!}\cdot\lim_{n\to\infty}\sum_{k=0}^{n}\int_0^1[(1-x)^p\cdot x^k]\cdot dx$$

$$=\frac{1}{p!}\cdot\lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}[(1-x)^p\cdot x^k]\cdot dx=\frac{1}{p!}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^p\cdot\sum_{k=1}^{n}x^k\right]\cdot dx$$

$$=\frac{1}{p!}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^p\cdot\frac{1-x^n}{1-x}\right]\cdot dx=\frac{1}{p!}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^{p-1}\cdot(1-x^n)\right]\cdot dx$$

$$=\frac{1}{p!}\cdot\int_0^1(1-x)^{p-1}\cdot dx-\frac{1}{p!}\cdot\lim_{n\to\infty}\int_0^1\left[(1-x)^{p-1}\cdot x^n\right]\cdot dx$$

$$=\frac{1}{p!}\cdot\int_0^1\left(C_{p-1}^0-x\cdot C_{p-1}^1+x^2\cdot C_{p-1}^2-\dots+(-1)^{p-2}\cdot x^{p-2}\cdot C_{p-1}^{p-2}+(-1)^{p-1}\cdot x^{p-1}\cdot C_{p-1}^{p-1}\right)\cdot dx$$

$$=\frac{1}{p!}\cdot\left(C_{p-1}^0-\frac{1}{2}\cdot C_{p-1}^1+\frac{1}{3}\cdot C_{p-1}^2-\dots+(-1)^{p-2}\cdot\frac{1}{p-1}\cdot C_{p-1}^{p-2}+(-1)^{p-1}\cdot\frac{1}{p}\cdot C_{p-1}^{p-1}\right),$$

since:

$$\lim_{n\to\infty}\int_0^1[(1-x^{p-1})\cdot x^n]\cdot dx=0.$$

III. Conclusions

Now we can say that we have solved the problem of calculating the limits of the sequences of type (H1.p). I worked inductively, starting from small values of p, q and k. The attentive and interested reader of these issues will notice that one can also work deductively. On the other hand, teachers, in class, or training circles / camps, with students capable of performance, can take other values for these numbers, only then the calculations will be more complicated and the results more beautiful, more spectacular.

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